arXiv:1010.0356v1 [math.AP] 2 Oct 2010

On the singular Q-curvature type equation

Mohammed Benalili

Abstract. This paper is devoted to the Q-curvature type equation with singularities; mainly we give existence and regularity results of solutions. To have positive solutions which will be meaningfully in conformal geometry we restrict ourself to special manifolds.

1. Introduction

In 1983, Paneitz [**16**] introduced a conformal fourth order operator defined on 4-dimensional Riemannian manifolds by

$$P_g^4(u) = \Delta_g^2 u - div_g\left(\frac{2}{3}R_g g - 2Ric_g\right)du$$

where $\Delta_g u = -div_g(\nabla_g u)$ is the Laplacian of u with respect to g, R_g is the scalar curvature with respect to g and, Ric_g is the Ricci curvature of g .

Branson [**6**] generalized the notion to manifolds of dimension $n \geq 5$ by letting

$$P_g^n(u) = \Delta_g^2 u - div_g\left(\frac{(n-2)^2+4}{2(n-1)(n-2)}R_g g - \frac{4}{n-2}Ric_g\right)du + \frac{n-4}{2}Q_g^n u$$

where

$$Q_g^n = \frac{1}{2(n-1)}\Delta_g R_g + \frac{n^3-4n^2+16n-16}{8(n-1)^2(n-2)^2}R_g^2 - \frac{2}{(n-2)^2}\left|Ric_g\right|^2.$$

Both operators P_g^4 and P_g^n are conformal operators that means that: for all $u \in C^\infty(M)$, $P_{\widetilde{g}}^4(\varphi u) = e^{-4\varphi}P_g^4(u)$ where $\widetilde{g} = e^{2\varphi}g$, while $P_g^n(\varphi u) = \varphi^{\frac{n+4}{n-4}}P_{\widetilde{g}}^n(u)$ where $\widetilde{g} = \varphi^{\frac{n+4}{n-4}}g$ and $n \geq 5$, $\varphi \in C^\infty(M)$. P_g^4 is the analogous of Δ_g in dimension 2 while P_g^n is the analogous of the conformal Laplacian $L_g u = \Delta_g u + \frac{n-2}{4(n-1)}R_g u$.

Fourth order equations of Sobolev growth have been the subject of intensive investigation the last tree decades because of theirs applications to conformal geometry, in particular to Paneitz-Branson operators; we refer the reader to [**2**], [**3**], [**4**], [**5**], [**7**], [**10**], [**11**], [**12**], [**13**], and [**16**] . These equations are also interesting because of theirs analogues in the second order which give applications to the conformal Laplacian. Recently Madani [**17**], has considered the Yamabé problem with

2000 *Mathematics Subject Classification.* Primary 58J05.

Key words and phrases. Q-curvature type equation, Hardy inequality, Singularities, regularity.

singularities which he solved under some geometric conditions. In this work we deal with the Q-curvature type equation with singularities which is of the form

$$\Delta^2 u - \nabla^i \left(a(x)\nabla_i u\right) + b(x)u = f\,|u|^{N-2}\,u \tag{1.1}$$

where the function $a(x)$ and $b(x)$ have singularities of order 2γ and 4α respectively, with γ, $\alpha \in (0,1)$ and $N = \frac{2n}{n-4}$ is the Sobolev critical exponent of the embedding $H_2^2(M) \hookrightarrow L^N(M)$. Mainly we give an existence result of solutions. Because of the lack of a maximum principle for fourth order elliptic equations, finding positive solutions is a hard question in the general case so to have positive solutions which will be meaningfully in conformal geometry we restrict ourself to special manifolds. Our results state as follows

Theorem 1. *Let (M,g) be a compact n-dimensional Riemannian manifold, $n \geq 6$, $a \in L^s(M)$, $b \in L^p(M)$, with $s > \frac{n}{2}$, $p > \frac{n}{4}$, $f \in C^\infty(M)$ a positive function and $P \in M$ such that $f(P) = \max_{x\in M} f(x)$.*

We suppose for $n > 6$

$$R_g(P) + \frac{4-n}{2n(n-2)(n^2-2n-4)}\frac{\Delta f(P)}{f(P)} > 0$$

and for $n = 6$,

$$R_g(P) > 0.$$

Then the equation (1.1) has a solution of class $C^{3-\left(\frac{p}{n}-E(\frac{p}{n})\right),\beta}$, $\beta \in \left]0, 1-\left(\frac{p}{n}-E(\frac{p}{n})\right)\right[$ where $E\left(\frac{p}{n}\right)$ denotes the entire part of $\frac{p}{n}$.

For $P \in M$, we define the function ρ on M by

$$\rho(Q) = \begin{cases} d(P,Q) & \text{if} \quad d(P,Q) < \delta(M) \\ \delta(M) & \text{if} \quad d(P,Q) \geq \delta(M) \end{cases} \tag{1.2}$$

where $\delta(M)$ denotes the injectivity radius of M.

For real numbers γ and α, consider the equation

$$\Delta^2 u - \nabla^\mu\left(\frac{a}{\rho^\gamma}\nabla_\mu u\right) + \frac{Q_g u}{\rho^\alpha} = f\,|u|^{N-2}\,u \tag{1.3}$$

and let $A = \left\{u \in H_2(M) : \int_M f\,|u|^N dv_g = 1\right\}$.

As Corollary of Theorem 1, we have

Corollary 1. *Let $\gamma \in (0,2)$, $\alpha \in (0,4)$, if we suppose*

for $n > 6$

$$R_g(P) + \frac{n^2-2n-4}{2n(n-1)}a(P) + \frac{4-n}{2n(n-2)(n^2-2n-4)}\frac{\Delta f(P)}{f(P)} > 0$$

and for $n = 6$,

$$R_g(P) > -3a\left(P\right).$$

Then the equation (1.3) has a solution of class $C^{3-\left(\frac{p}{n}-E(\frac{p}{n})\right),\beta}$, $\beta \in \left]0, 1-\left(\frac{p}{n}-E(\frac{p}{n})\right)\right[$.

In the sharp case $\gamma = 2$, $\alpha = 4$, let γ,α as in section 4, we get

Theorem 2. *Suppose that* $1+Q_g(P)K(n,2,-4)^2>0$ *and* $Q_{2,4}(M)(K(n,2)^2\left\|f\right\|_\infty^{\frac{2}{N}}<1+Q_g(P)K(n,2,-4)^2$*, then the equation*

$$\Delta^2u-\nabla^\mu(\frac{a}{\rho^2}\nabla_\mu u)+\frac{Q_g u}{\rho^4}=f\left|u\right|^{N-2}u$$

has a non trivial solution.

When (M,h) is a compact flat manifold, let $g=Ah$ where $A=e^{-\rho^{2-\rho}}$ and $\alpha\in(1,2)$ and ρ is given by (1.2), we give a geometric interpretation of ours results

Theorem 3. *Suppose that the manifold* (M,h) *is a compact flat of dimension* $n\geqslant 6$ *and consider the coformal metric* $g=Ah$*, where* $A=e^{-\rho^{2-\alpha}}$ *,* $\alpha\in(1,2)$ *and* ρ *sufficient small. Let* f *be a* C^∞ *positive function on* M *and* $P\in M$ *such that* $f(P)=\max_{x\in M}f(x)$*.*

We suppose for $n>6$

$$R_g(P)+\frac{4-n}{2n(n-2)(n^2-2n-4)}\frac{\Delta f(P)}{f(P)}>0$$

then there exists a metric $\widetilde{g}$ *conformal to* g *and of class* $C^{3-\left(\frac{p}{n}-E(\frac{p}{n})\right),\beta}$*,* $\beta\in(0,1)$ *where* p *is as in Theorem 1, such that* f *is the* Q*-curvature of the manifold* $(M,\widetilde{g})$*.*

Our paper is organized as follows: in the first section, we give an Hardy inequality on compact manifolds, in the second one we establish the regularity of the Paneitz-Branson operator which leads us to construct the Green function to the Schrödinger biharmonic operator this latter allows us to obtain a priori estimates to a solution of some biharmonic equation. The third section is devoted to the study of the Q-curvature equation with singularities of order $0<\gamma<2$ and $0<\alpha<4$. In the fourth section we consider the sharp singularities i.e. $\gamma=2$ and $\alpha=4$. In the last section, we give an interpretation in conformal geometry.

2. Hardy inequality on compact manifolds

Let (M,g) be a compact n-dimensional Riemannian manifold. We consider the space $L^p(M,\rho^\gamma)$, $1\leq p\leq\infty$, of measurable functions u on M such that

$$\left\|u\right\|_{p,\rho^\gamma}^p=\int_M\rho^\gamma\left|u\right|^p dv_g<+\infty \tag{2.1}$$

where ρ is the function defined above and $\gamma\in R$.

The space $L^p(M,\rho^\gamma)$, endowed with the norm $\left\|u\right\|_{p,\rho^\gamma}^p$, is a Banach space and we have

Lemma 1. *(Hardy inequality)*

For any function $u\in C_o^\infty(R^n)$*, there exists a constant* $C>0$ *such that*

$$\left\|\left|x\right|^{\frac{\gamma}{p}}u\right\|_p\leq C\left\|\left|x\right|^\beta\nabla^l u\right\|_q \tag{2.2}$$

where p*,* q *and* γ *real numbers such that*

$$1\leq q\leq p\leq\frac{nq}{n-lq},\ n>lq,\ \frac{\gamma}{p}=\beta-l+n\left(\frac{1}{q}-\frac{1}{p}\right)>-\frac{n}{p}.$$

The optimal constant in 2.2 will be denoted by $K(n,l,\gamma,\beta)$ and $K(n,l,\gamma)$ when $\beta=0$.

From the above lemma, we obtain

Lemma 2. *Let (M,g) be a compact n- dimensional Riemannian manifold, $n\geq 5$, and p, q and γ real numbers satisfying*

$$1\leq q\leq p\leq \frac{nq}{n-2q},\ n>2q,\ \frac{\gamma}{p}=-2+n\left(\frac{1}{q}-\frac{1}{p}\right)>0.$$

For any $\varepsilon>0$, there is a constant $A(\varepsilon,q,\gamma)$ such that

$$\forall f\in H_2^q(M),\quad \|f\|_{p,\rho^\gamma}^q\leq(1+\varepsilon)\,K^q(n,2,\gamma)\left\|\nabla^2 f\right\|_q^q+A(\varepsilon,q,\gamma)\,\|f\|_q^q\,.$$

Proof. Let $\{B_i\}_{1\leq i\leq m}$ be a finite covering of M by geodesic balls of small radius $\delta>0$, $\{(B_i,\varphi_i)\}_i$, where $\varphi_i=\exp_{p_i}^{-1}$, is an associated atlas and $(\eta_i)_{1\leq i\leq m}$ is a partition of unity subordinated to the covering $\{B_i\}_{1\leq i\leq m}$.

Let $f\in C_o^\infty(M)$

$$\|f\|_{p,\rho^\gamma}^q=\||f|^q\|_{\frac{p}{q},\rho^\gamma}=\left\|\sum_{i=1}^m a_i\,|f|^q\right\|_{\frac{p}{q},\rho^\gamma}\leq\sum_{i=1}^m\left\|a_i^{\frac{1}{q}}f\right\|_{p,\rho^\gamma}^q.$$

The function $\widetilde{f}=(a_if)\,o\varphi_i^{-1}$ may be considered as a function defined on R^n by extending it by 0 outside the support, so applying inequality (2.2), with $\beta=0$ and $l=2$ to $\widetilde{f}$, we get

$$\left(\int_{R^n}|x|^\gamma\left|\widetilde{f}\right|^p dx\right)^{\frac{1}{p}}\leq K\left(n,2,\gamma\right)\left(\int_{R^n}\left|\nabla^2\widetilde{f}\right|^q dx\right)^{\frac{1}{q}}. \tag{2.3}$$

Now, we have to express the derivatives of the function $\widetilde{f}$ in terms of the Euclidean derivatives. If the coordinate system is normal at a point $P\in M$, the expansion of the metric tensor at P writes as Q

$$g_{ij}\left(Q\right)=\delta_{ij}+O\left(\rho^2\right)$$

where $\rho=d(P,Q)<\delta(M)$, and the expansions of the Christoffel symbols are of the form

$$\Gamma_{ij}^k(Q)=O(\rho).$$

Now since

$$\nabla_{ij}\widetilde{f}\left(Q\right)=\frac{\partial^2\widetilde{f}}{\partial x_i\partial x_j}\left(Q\right)-\Gamma_{ij}^s\left(Q\right)\frac{\partial\widetilde{f}}{\partial x_s}\left(Q\right)$$

we obtain that

$$g^{ik}\left(Q\right)g^{jl}\left(Q\right)\nabla_{ij}\widetilde{f}\left(Q\right)\nabla_{kl}\widetilde{f}\left(Q\right)\leq\left(1+O\left(\delta^2\right)\left|\nabla_E^2\widetilde{f}\left(Q\right)\right|^2\right)$$

$$+\left|\nabla_E^2\widetilde{f}\left(Q\right)\right|\left|\nabla_E\widetilde{f}\left(Q\right)\right|O\left(\delta\right)+\left|\nabla_E\widetilde{f}\left(Q\right)\right|^2O\left(\delta^2\right).$$

To estimate the rectangular term, we use the inequality

$$ab\leq\eta a^2+\frac{b^2}{4\eta}$$

valid for any positive real numbers a, b, η and get for any $\varepsilon>0$, there is a constant $C(\varepsilon)$ such that

$$\left|\nabla^2 \widetilde{f}(Q)\right|^2 \leq (1+\varepsilon)\left|\nabla_E^2 \widetilde{f}(Q)\right|^2 + C(\varepsilon)\left|\nabla_E \widetilde{f}(Q)\right|^2 \tag{2.4}$$

and similarly, we obtain

$$\left|\nabla_E^2 \widetilde{f}(Q)\right|^2 \leq (1+\varepsilon)\left|\nabla^2 \widetilde{f}(Q)\right|^2 + C(\varepsilon)\left|\nabla \widetilde{f}(Q)\right|^2. \tag{2.5}$$

Since M is compact there exist constants λ, μ with

$$0 < \lambda \leq \sqrt{|g|} \leq \mu.$$

Consequently

$$I_i = \int_M \rho^\gamma \left|a_i^{\frac{1}{q}} f\right|^p dv_g = \int_{R^n} |x|^\gamma \left|\widetilde{f}\right|^p \sqrt{|g|}dx \leq \mu \int_{R^n} |x|^\gamma \left|\widetilde{f}\right|^p dx$$

where $|x| = d(P,Q) < \delta(M)$ denotes the injectivity radius and $Q = \exp_P x$. So by the inequality (2.3) we write

$$I_1 \leq \mu K(n,2,\gamma)^p \left(\int_{R^n} \left|\nabla_E^2 \widetilde{f}\right|^q dx\right)^{\frac{p}{q}}.$$

Taking account of the inequality (2.5) and of the following inequality, let a, b, $s \geq 1$ be positive real numbers, for any $\varepsilon_1 > 0$, there is a constant $C(\varepsilon_1, s)$ such that

$$(a+b)^s \leq (1+\varepsilon_1) a^s + C(\varepsilon_1, s) b^s \tag{2.6}$$

we obtain

$$I_i^{\frac{q}{p}} \leq \mu^{\frac{q}{p}} K(n,2,\gamma)^q \left(\int_{R^n} \left|\nabla_E^2 \widetilde{f}\right|^q dx\right)$$

$$\leq \left(\mu^{\frac{q}{p}} \lambda^{-1}\right) K(n,2,\gamma)^q \left\{(1+\varepsilon)\left\|\nabla^2 \widetilde{f}\right\|_q^q + C(\varepsilon,q,\gamma)\left\|\nabla \widetilde{f}\right\|_q^q\right\}.$$

Now since $\widetilde{f} = a_i^{\frac{1}{q}} f$, there exits a constant $c > 0$ such that

$$\left|\nabla \widetilde{f}\right| \leq c|f| + a_i^{\frac{1}{q}} |\nabla f|$$

and

$$\left|\nabla^2 \widetilde{f}\right| \leq c(|f| + |\nabla f|) + a_i^{\frac{1}{q}} \left|\nabla^2 f\right|.$$

From the interpolation formula (see [**1**] page 93), for any $\eta > 0$, there is a constant $C(\eta)$ such that

$$\|\nabla f\|_q^q \leq \eta \left\|\nabla^2 f\right\|_q^q + C(\eta)\|f\|_q^q \tag{2.7}$$

applying the inequality (2.6) and letting the injectivity radius small enough so that λ and μ are close to 1, we get, for any $\varepsilon > 0$ there is a constant $A(\varepsilon,q,\gamma)$

$$\|f\|_{p,\rho^\gamma}^q \leq (1+\varepsilon) K(n,2,\gamma)^q \left\|\nabla^2 f\right\|_q^q + A(\varepsilon,q,\gamma)\|f\|_q^q.$$

□

As a corollary of Lemma 2, we have the following Sobolev inequality

Lemma 3. *Let (M,g) be a compact n- dimensional Riemannian manifold, $n \geq 5$, and p, γ real numbers satisfying*

$$2 \leq p \leq \frac{2n}{n-4}, \ \frac{\gamma}{p} = -2 + n\left(\frac{1}{2} - \frac{1}{p}\right).$$

For any $\varepsilon > 0$, there is a constant $A(\varepsilon, q, \gamma)$ such that

$$\forall f \in H_2(M), \quad \|f\|_{p,\rho^\gamma}^2 \leq (1+\varepsilon)\, K(n,2,\gamma)^2 \,\|\Delta f\|_2^2 + A(\varepsilon,\gamma)\,\|f\|_2^2 \,.$$

In fact, it is known on compact manifold (see [**1**] page 115) that there is a constant $c > 0$ such that

$$\left\|\nabla^2 f\right\|_2^2 \leq \|\Delta f\|_2^2 + c\,\|\nabla f\|_2^2$$

and our formula follows from the interpolation formula (2.7).

Now we derive a Kondrakov type result

Lemma 4. *Let (M,g) be a compact n- dimensional Riemannian manifold and p, q, $\gamma < 0$ real numbers such that*

$$1 \leq q \leq p \leq \frac{nq}{n-2q}$$

If $\frac{\gamma}{p} = -2 + n\left(\frac{1}{q} - \frac{1}{p}\right)$, the inclusion $H_2^q(M) \subset L^p(M, \rho^\gamma)$ is continuous.

If $\frac{\gamma}{p} > -2 + n\left(\frac{1}{q} - \frac{1}{p}\right)$, the inclusion $H_2^q(M) \subset L^p(M, \rho^\gamma)$ is compact.

Proof. The first part of Lemma 4 is a consequence of Lemma 2. To prove the second part of Lemma 4, we consider the following inclusions $H_2^q(M) \subset L^r(M) \subset L^p(M, \rho^\gamma)$ and we have to show that the first inclusion is compact and the second one is continuous. By the Kondrakov's theorem we must have

$$\frac{1}{r} > \frac{1}{q} - \frac{2}{n}. \tag{2.8}$$

The Hölder inequality allows us to write

$$\int_M \rho^\gamma |u|^p \, dv_g \leq \left(\int_M |u|^r \, dv_g\right)^{\frac{p}{r}} \left(\int_M \rho^{\gamma r'} dv_g\right)^{\frac{1}{r'}} \tag{2.9}$$

with $r' = \frac{r}{r-p}$ and $r > p$.

The second integral in the right-hand side of (2.9) will be convergent if

$$\gamma r' = \frac{\gamma r}{n-p} > -n \quad \text{i.e.} \quad \frac{1}{r} < \frac{\gamma+n}{np}$$

so by (2.9) we must have

$$\frac{1}{q} - \frac{2}{n} < \frac{\gamma+n}{np}$$

i.e.

$$\frac{\gamma}{p} > -2 + n\left(\frac{1}{q} - \frac{1}{p}\right).$$

□

Now, we quote the following Lemma due to Djadli-Hebey-Ledoux [**10**] and improved by Hebey [**12**] which will be used in the sequel of this paper.

Lemma 5. *Let M be a Riemannian compact manifold with dimension $n \geq 5$. For any $\epsilon > 0$ there is a constant $A_2(\epsilon)$ such that for any $u \in H_2$, $\|u\|_N^2 \leq (1+\epsilon)K_2^2 \|\Delta u\|_2^2 + A_2(\epsilon)\|u\|_2^2$, where K_2 is the best Sobolev constant.*

3. Regularity of the Paneitz-Branson operator

The Green function of the Laplacian operator $G : M \times M \to R$, $(P,Q) \to G(P,Q)$ is defined as the solution in the distribution sense to the equation

$$\Delta G(P,.) = \delta_P - \frac{1}{V(M)} \tag{3.1}$$

where $V(M)$ is the Riemannian volume of M. So for any $\varphi \in C^\infty(M)$, we have

$$\Delta\varphi(P) = \int_M G(P,Q)\Delta^2\varphi(Q)\, dv_g(Q). \tag{3.2}$$

Multiplying (3.2) by $G(Q,.)$ and integrating over M, we get

$$\varphi(P) = \int_M \int_M G(P,Q)G(Q,R)\Delta^2\varphi(R) dv_g(Q)\, dv_g(R) + \frac{1}{V(M)}\int_M \varphi(Q) dv_g(Q) \tag{3.3}$$

and letting

$$G_2(P,Q) = \int_M G(P,R)G(R,Q)dv_g(R)$$

(3.3) becomes

$$\varphi(P) = \int_M G_2(P,Q)\Delta^2\varphi(Q)dv_g(Q) + \frac{1}{V(M)}\int_M \varphi(R)dv_g(R). \tag{3.4}$$

According to Giraud's lemma [**14**],

$$|G_2(P,Q)| \leq Cd(P,Q)^{4-n} \quad \text{if } n > 4.$$

Hence G_2 is the green function of the bi-harmonic operator.

Suppose now that $b \in L^p(M)$.

We put $\Gamma_o(P,Q) = G_2(P,Q)$ and define recursively for $j \geqslant 1$,

$$\Gamma_j(P,Q) = -\int_M \Gamma_{j-1}(P,R)b(R)\Gamma_o(R,Q)dv_g(R)$$

Γ_j is well defined and by Giraud's lemma [**14**]

$$|\Gamma_j(P,Q)| \leq \begin{cases} C_j \|b\|_p^j d(P,Q)^{(j+1)(4-n)+jn\left(1-\frac{1}{p}\right)} & \text{if } (j+1)\frac{p}{p+j} < \frac{n}{4} \\ C_j \|b\|_p^j (1+|\log d(P,Q)|) & \text{if } (j+1)\frac{p}{p+j} = \frac{n}{4} \\ C_j \|b\|_p^j & \text{if } (j+1)\frac{p}{p+j} > \frac{n}{4} \end{cases}.$$

For $Q \in M$, we consider a function $u_Q \in C^4(M)$, $\gamma \in (0,1)$which will be determinated later and define, for $m > \frac{n}{4}$

$$H(Q,.) = \Gamma_o(Q,.) + \sum_{j=1}^m \Gamma_j(Q,.) + u_Q.$$

Since $p > \frac{n}{4}$, we have

$$\|\Gamma_j(Q,.)\|_1 \leq C_j \|b\|_p^j \left\| d(P,Q)^{(j+1)(4-n)+jn\left(1-\frac{1}{p}\right)} \right\|_1 < +\infty$$

hence $H(Q,.)\in C^4(M-\{Q\})\cap L^1(M)$.

$H(Q,.)$ will be a Green function to $P(u)=\Delta^2 u-\nabla^\mu(a\nabla_\mu u)+bu$ on M if for every $\varphi\in C^4(M)$, $H(Q,.)$ solves the equation

$$\varphi(Q)=\int_M H(Q,R)P(\varphi)(R)dv_g(R)+\frac{1}{V(M)}\int_M \varphi(R)\,dv_g(R)$$

$$=\int_M H(Q,R)\Delta^2\varphi(R)dv_g(R)-\int_M H(Q,R)\left(\nabla^\mu\left(a\nabla_\mu\varphi\right)\right)(R)dv_g(R)+$$

$$\int_M H(Q,R)b(R)\varphi(R)dv_g(R)+\frac{1}{V(M)}\int_M\varphi(R)\,dv_g(R)$$

$$=\int_M \Delta_R^2 H(Q,R).\varphi(R)dv_g(R)-\int_M\left(\nabla^\mu\left(a\nabla_\mu H(Q,R)\right)\right)(R).\varphi(R)dv_g(R)$$

$$+\int_M H(Q,R)b(R)\varphi(R)dv_g(R)+\frac{1}{V(M)}\int_M\varphi(R)\,dv_g(R)\,. \tag{3.5}$$

The first integral of (3.5) reads

$$\int_M \Delta_R^2 H(Q,R).\varphi(R)dv_g(R)=\int_M\Delta_R^2\Gamma_o(Q,R)\varphi(R)dv_g(R)$$

$$+\sum_{j=1}^m\int_M\Delta_R^2\Gamma_j(Q,R).\varphi(R)dv_g(R)+\int_M\Delta_R^2 u_Q(R).\varphi(R)\,dv_g(R)$$

$$=\varphi(Q)-\frac{1}{V(M)}\int_M\varphi(R)\,dv_g(R)-\sum_{j=1}^m\int_M b(R)\Gamma_{j-1}(Q,R)\varphi(R)\,dv_g(R) \tag{3.6}$$

$$+\int_M\Delta^2 u_Q(R)\varphi(R)\,dv_g(R)+\frac{1}{V(M)}\int_M\varphi(R)dv_g(R)\sum_{j=1}^m\int_M\Gamma_{j-1}(Q,S)\,b(S)dv_g(S)$$

and plugging (3.6) in (3.5), we obtain

$$\int_M H(Q,R)P(\varphi)(R)dv_g(R)=\varphi(Q)-\frac{1}{V(M)}\int_M\varphi(R)\,dv_g(R)$$

$$-\int_M a(R)\Gamma_m(Q,R)\varphi(R)\,dv_g(R)+\int_M\left(\Delta_R^2u_Q(R)-\left(\nabla^\mu\left(a\nabla_\mu u_Q\right)\right)(R)+bu_Q(R)\right)\varphi(R)\,dv_g(R)$$

$$-\sum_{j=0}^m\int_M\left(\nabla^\mu\left(a\nabla_\mu\Gamma_j(Q,R)\right)\right)(R).\varphi(R)dv_g(R).$$

So solving the equation (3.5) is equivalent to solve the following equation

$$\Delta_R^2u_Q(R)-\left(\nabla^\mu\left(a\nabla_\mu u_Q\right)\right)(R)+bu_Q(R)=a(R)\Gamma_m(Q,R)+\sum_{j=0}^m\left(\nabla^\mu\left(a\nabla_\mu\Gamma_j(Q,.)\right)\right)(R)\ . \tag{3.7}$$

Consider the functional T_Q defined on $H_2(M)$ by

$$T_Q(u)=\int_M\left(a(R)\Gamma_m(Q,R)u(R)-\sum_{j=0}^m a(R)\,\nabla_R\Gamma_j(Q,R)\bigtriangledown u(R)\right)dv_g$$

It obvious that the functional T_Q is continuous on the Hilbert space $H_2(M)$, so since the operator $u \to \int_M uP(u)\,dv_g$ is coercive it follows by Lax-Milgram theorem that there is a unique $u_Q \in H_2$ such that for any $\varphi \in H_2$

$$\int_M \varphi P(u_Q)\,dv_g = T_Q(\varphi)$$

that is to say u_Q is a weak solution of the equation (3.7) and by classical regularity arguments, we get that $u_Q \in C^{4,\gamma}(M)$ where $\gamma \in (0,1)$.

Next let $U \subset M$ be an open set.

Lemma 6. *Let $h \in L^1_{loc}(U)$, $\xi \in C_o^\infty(U)$. If $u \in H^2_{2,loc}(U)$ is a weak solution of the equation $\Delta^2 u - \nabla^i(a\nabla_i u) + bu = h$, then*

$$(\xi u)(Q) = \int_M H(Q,R)P(u)(R)\xi(R)\,dv_g(R)$$

$$+\int_M H(Q,R)u(R)\left\{\Delta^2\xi(R) - \nabla^\mu(a\nabla_\mu\xi)(R)\right\}dv_g(R)$$

$$+2\int_M u\left(\Delta H(Q,R)\Delta\zeta(R) + H(Q,R)\Delta^2\xi(R) - 2\left\langle\nabla H(Q,R),\nabla\Delta\xi(R)\right\rangle(Q)\right)dv_g(R)$$

$$+2\int_M u(R)\left(\left\langle\nabla H(Q,R),\nabla(\Delta\xi)+\Delta(\nabla\xi)\right\rangle + H(Q,R)\left(\nabla^\mu\Delta(\nabla_\mu\xi) - \Delta^2\xi\right)\right)(R)$$

$$-2\int_M u(R)\left(\Delta H(Q,R)\Delta\xi(R) + 2\left\langle\nabla^2\xi(R),\nabla^2 H(Q,R)\right\rangle + 2\left\langle\nabla H(Q,R),\Delta(\nabla\xi(R))\right\rangle\right)$$

$$+\int_M u(R)\left\langle\nabla\xi(R),\nabla(\Delta H(Q,R)) + \Delta(\nabla H(R,Q)\right\rangle dv_g(R)$$

$$+\frac{1}{V(M)}\int_M(\xi u)(Q)dv_g(Q).$$

Proof. Let $(u_n) \subset C_o^\infty(U)$ such that $u_n \to u$ in $H^2_{2,loc}(U)$. We extend the functions ξ, u_n by 0 outside U to have a functions defined on all M and by applying the formula (3.1), we get

$$(\xi u_n)(Q) = \int_M H(Q,R)P(\xi u_n)(R)dv_g(R) + \frac{1}{V(M)}\int_M(\xi u_n)(Q)dv_g(Q). \tag{3.8}$$

Now we compute

$$P(\xi u_n) = \Delta^2(\xi u_n) - \nabla^\mu(a\nabla_\mu\xi u_n) + b\xi u_n$$
$$= \xi\Delta^2(u_n) + u_n\Delta^2(\xi) + 2\Delta u_n\Delta\xi - 2\left\langle\Delta\nabla u_n,\nabla\xi\right\rangle - 2\left\langle\nabla u_n,\Delta\nabla\xi\right\rangle$$
$$+2\left\langle\nabla^2 u_n,\nabla^2\xi\right\rangle - 2\left\langle\nabla u_n,\nabla\Delta\xi\right\rangle - 2\left\langle\nabla\xi,\nabla\Delta u_n\right\rangle$$
$$-\xi\nabla^\mu(a\nabla_\mu u_n) - u_n\nabla^\mu(a\nabla_\mu\xi) - 2a\left\langle\nabla u_n,\nabla\xi\right\rangle + b\xi u_n.$$

Consequently

$$(\xi u_n)(Q) = \int_M \xi(R)H(Q,R)P(u_n)(R)dv_g(R)$$

$$+\int_M H(Q,R)u_n(R)\left\{\Delta^2\xi(R) - \nabla^\mu(a\nabla_\mu\xi)(R)\right\}dv_g(R)$$

$$+\int_M 2H(Q,R)\left(\Delta\xi\Delta u_n - \left\langle\nabla(\Delta\xi)+\Delta(\nabla\xi),\nabla u_n\right\rangle\right)dv_g(R)$$

$$-\int_M 2H(Q,R)\left\langle\nabla\xi,\nabla(\Delta u_n)+\Delta(\nabla u_n)\right\rangle dv_g(Q)\,dv_g(Q)$$

$$-\int_M 2H(Q,R)\left(a(R)\langle\nabla u_n,\nabla\xi\rangle+2\left\langle\nabla^2 u_n,\nabla^2\xi\right\rangle\right)dv_g(R)$$
$$+\frac{1}{V(M)}\int_M(\xi u_n)(Q)dv_g(Q).$$

Also we have

$$\int_M 2H(Q,R)\left(\Delta\zeta.\Delta u_n-\langle\nabla(\Delta\xi)+\Delta(\nabla\xi),\nabla u_n\rangle\right)dv_g(R)=$$
$$2\int_M u_n\left(\Delta H(Q,R)\Delta\zeta(R)+H(Q,R)\Delta^2\xi(R)-2\langle\nabla H(Q,R),\nabla(\Delta\xi)\rangle\right)dv_g(R)$$
$$+2\int_M u_n\left(\langle\nabla H(Q,R),\nabla(\Delta\xi)+\Delta(\nabla\xi)\rangle+H(Q,R)\left(\nabla^\mu\Delta(\nabla_\mu\xi)-\Delta^2\xi\right)\right)(R)$$

so

$$\int_M 2H(Q,R)\langle\nabla\xi,\nabla(\Delta u_n)+\Delta(\nabla u_n)\rangle dv_g(Q)dv_g(Q)=$$
$$=2\int_M u_n(R)\left(\Delta H(Q,R)\Delta\xi(R)+2\left\langle\nabla^2\xi(R),\nabla^2H(Q,R)\right\rangle\right)dv_g$$
$$-4\int_M u_n(R)\langle\nabla H(Q,R),\Delta(\nabla\xi(R))\rangle dv_g$$
$$-\int_M u_n(R)\langle\nabla\xi(R),\nabla(\Delta H(Q,R))+\Delta(\nabla H(R,Q)\rangle dv_g(R)$$

Consequently

$$(\xi u_n)(Q)=\int_M H(Q,R)P(u_n)(R)\xi(R)dv_g(R)$$
$$+\int_M H(Q,R)u_n(R)\left\{\Delta^2\xi(R)-\nabla^\mu(a\nabla_\mu\xi)(R)\right\}dv_g(R)$$
$$+2\int_M u_n\left(\Delta H(Q,R)\Delta\zeta(R)+H(Q,R)\Delta^2\xi(R)-2\langle\nabla H(Q,R),\nabla(\Delta\xi)\rangle(Q)\right)dv_g(R)$$
$$+2\int_M u_n\left(\langle\nabla H(Q,R),\nabla(\Delta\xi)+\Delta(\nabla\xi)\rangle+H(Q,R)\left(\nabla^\mu\Delta(\nabla_\mu\xi)-\Delta^2\xi\right)\right)(R)$$
$$-2\int_M u_n(R)\left(\Delta H(Q,R)\Delta\xi(R)+2\left\langle\nabla^2\xi(R),\nabla^2H(Q,R)\right\rangle+2\langle\nabla H(Q,R),\Delta(\nabla\xi(R))\rangle\right)$$
$$+\int_M u_n(R)\langle\nabla\xi(R),\nabla(\Delta H(Q,R))+\Delta(\nabla H(R,Q)\rangle dv_g(R) \tag{3.9}$$
$$+\frac{1}{V(M)}\int_M(\xi u)_n(Q)dv_g(Q).$$

Now

$$\int_U|\xi(R)P(H(Q,.))(R)|dv_g(R)\le$$
$$C\sup_{x\in U}|\xi(R)|\sup_{Q\in U}\int_{B_r(Q)}d(Q,R)^{-n+2}dv_g(R)<+\infty.$$

Letting

$$F(Q)=\int_M\xi(R)H(Q,R)P(u)(R)dv_g(R)$$
$$=\int_M\xi(R)h(R)H(Q,R)dv_g(R)$$

and

$$F_n(Q) = \int_M \xi(R) H(Q,R) P(u_n)(R) dv_g(R)$$
$$= \int_M \xi(R) H(Q,R) h_n(R) dv_g(R)$$

with

$$h_n = P(u_n)$$

we infer that

$$\int_U (F_n - F)(Q) dv_g(Q) = \int_U \int_M \xi(R) H(Q,R)(h_n - h)(R) dv_g(R)$$

which goes to 0 as n goes to $+\infty$.

In the same way if we put

$$G(Q) = \int_M H(Q,R) u(R) \left\{ \Delta^2 \xi(R) - \nabla^\mu (a \nabla_\mu \xi)(R) \right\} dv_g(R)$$

we obtain

$$\int_M \left| H(Q,R) \left\{ \Delta^2 \xi(R) - \nabla^\mu (a \nabla_\mu \xi)(R) \right\} \right| dv_g(R) \leq$$

$$\left\| \Delta^2 \xi \right\|_\infty \sup_{Q \in M} \int_M d(Q,R)^{-n+4\beta} dv_g(R) + \|\nabla \xi\|_\infty \|a\|_p \|\nabla_R H\|_{\frac{p}{p-1}} < +\infty$$

then

$$\int_U |G(Q)| dv_g(Q) \leq$$

$$\sup_{Q \in U} \left| \int_U \left(H(Q,R) \Delta^2 \xi(R) + a(R) \langle \nabla H(Q,R), (\nabla \xi)(R) \rangle \right) dv_g(R) \right| \int_M |u(R)| dv_g(R) < +\infty.$$

Letting

$$G_n(Q) = \int_M H(Q,R) u_n(R) \left\{ \Delta^2 \xi(R) - \nabla^\mu (a \nabla_\mu \xi)(R) \right\} dv_g(R)$$

then

$$\int_U |G_n(Q)| dv_g(Q) < +\infty$$

and

$$\int_U |G_n(Q) - G(Q)| dv_g(Q) \leq$$

$$\sup_{Q \in U} \left| \int_U \left(H(Q,R) \Delta^2 \xi(R) + a(R) \langle \nabla H(Q,R), (\nabla \xi)(R) \rangle \right) dv_g(R) \right| \|u_n - u\|_{L^1(M)}.$$

Then $G_n \to G$ in $L^1(U)$ and almost every in U. The same is also true for the remaining terms of the right side hand of (3.9) and we get the required formula. □

Now we recall the definition of Kato-Stummel space,

Definition 1. *Kato-Stummel space $K^n(U)$ is defined as the space of measurable functions $f: U \to R$ such that for every $t > 0$,*

$$\varphi_f(t,U) = \sup_{Q \in M} \int_M \frac{|f(S)| \chi_U(S)}{d(Q,S)^l} dv_g(S) < +\infty$$

with $l = (j+1)(n-4) - jn\left(1 - \frac{1}{p}\right)$, $j \geq 1$ *any integer*

and

$$\lim_{t\to 0^+}\varphi_f\left(t,U\right)=0.$$

where U denotes an open set of M and χ_U is the characteristic function of U.

We consider the C^{∞}-function $\eta : R\to[0,1]$ with compact support given by

$$\eta(t)=\begin{cases} 1 \text{ for } |t|\leq\frac{1}{2} \\ 0 \quad\text{for } |t|\geq 1\end{cases}$$

and for any P, $Q\in M$, we put

$$\eta_\delta\left(P,Q\right)=\eta\left(\delta^{-1}d\left(P,Q\right)\right)$$

and for $R\in M$, $S\in U$, with $R\neq S$, we put

$$F_\delta\left(R,S\right)=\int_M\frac{\left|f\left(T\right)\right|\eta_\delta\left(S,T\right)\eta_\delta\left(R,T\right)}{d\left(T,S\right)^l d\left(T,R\right)^l}dv_g\left(T\right)$$

where $l=\left(j+1\right)\left(n-4\right)-jn\left(1-\frac{1}{p}\right)$, $j\geq 1$, any integer $0<\delta<d(S,\partial U)/4$ and $d\left(S,\partial U\right)$ is the distance from S to the boundary ∂U of U.

First we quote the following useful lemma [**17**]

Lemma 7. *There exists a constant $c\left(n\right)>0$ such that*
$F_\delta\left(R,S\right)\leq c\left(n\right)\varphi_f\left(\delta,B_{3\delta}\left(Q\right)\right)\eta_{4\delta}\left(R,S\right)d(R,S)^{-l}$.

Now, we give a representation formula to the solution of equation (1.1)

Lemma 8. *Let $Q_o\in U$ and $r_o>0$ such that $B_{2r_o}\subset U$. For almost every $Q\in B_{r_o}\left(Q_o\right)$*

$$\begin{aligned} u(Q)=&\int_M H(Q,R)h(R)\eta_{r_o}\left(Q,R\right)dv_g\left(R\right)\\ &+\int_M H(Q,R)u(R)\left\{\Delta^2\eta_{r_o}\left(Q,R\right)-\nabla^{\mu}\left(a\nabla_{\mu}\eta_{r_o}\left(Q,R\right)\right)\left(R\right)\right\}dv_g\left(R\right)\\ &+2\int_M u\left(R\right)\left(\Delta H(Q,R)\Delta\eta_{r_o}\left(Q,R\right)+H(Q,R)\Delta^2\eta_{r_o}\left(Q,R\right)\right)dv_g\left(R\right)\\ &-4\int_M u(R)\left\langle\nabla H\left(Q,R\right),\nabla\left(\Delta\eta_{r_o}\left(Q,R\right)\right)\right\rangle dv_g(R)\\ &-2\int_M u\left(R\right)\left\langle\nabla H\left(Q,R\right),\nabla\left(\Delta\eta_{r_o}\left(Q,R\right)\right)\right\rangle\left(Q\right)dv_g\left(R\right)\\ &+2\int_M u\left(R\right)\left(\left\langle\nabla H(Q,R),\nabla\left(\Delta\eta_{r_o}\left(Q,R\right)\right)+\Delta\left(\nabla\eta_{r_o}\left(Q,R\right)\right)\right\rangle\right)dv_g\left(R\right)\\ &+2\int_M u\left(R\right)H(Q,R)\left(\nabla^{\mu}\Delta\left(\nabla_{\mu}\eta_{r_o}\left(Q,R\right)\right)-\Delta^2\eta_{r_o}\left(Q,R\right)\right)dv_g\left(R\right)\\ &-2\int_M u(R)\left(\Delta H\left(Q,R\right)\Delta\eta_{r_o}\left(Q,R\right)+2\left\langle\nabla^2\eta_{r_o}\left(Q,R\right),\nabla^2 H(Q,R)\right\rangle\right)\\ &+4\int_M\left\langle\nabla H(Q,R),\Delta\left(\nabla\eta_{r_o}\left(Q,R\right)\right)\right\rangle dv_g(R)\\ &+\int_M u\left(R\right)\left\langle\nabla\eta_{r_o}\left(Q,R\right),\nabla\left(\Delta H\left(Q,R\right)\right)+\Delta\left(\nabla H(R,Q)\right\rangle dv_g\left(R\right)\\ &+\frac{1}{V(M)}\int_M\eta_{r_o}\left(Q,R\right)u(R)dv_g\left(R\right). \end{aligned}\tag{3.10}$$

PROOF. Let $S \in B_{r_o}(Q_o)$ and consider the function

$$\xi(Q) = \eta_{r_o}(S,Q)$$

then $\xi \in C_o^\infty(U)$ and

$$\eta_{r_o}(S,Q)\,u(Q) = \int_M H(Q,R)P(u)(R)\eta_{r_o}(S,R)\,dv_g(R)$$
$$+\int_M H(Q,R)u(R)\left\{\Delta^2\eta_{r_o}(S,R) - \nabla^\mu\left(a\nabla_\mu\eta_{r_o}(S,R)\right)\right\}dv_g(R)$$
$$+2\int_M u(R)\left(\Delta H(Q,R)\Delta\eta_{r_o}(S,R) + H(Q,R)\Delta^2\eta_{r_o}(S,R)\right)dv_g(R)$$
$$-4\int_M u(R)\left\langle\nabla H(Q,R),\nabla\left(\Delta\eta_{r_o}(S,R)\right)\right\rangle(Q)\,dv_g(R)$$
$$-2\int_M u(R)\left\langle\nabla H(Q,R),\nabla\left(\Delta\eta_{r_o}(S,R)\right)\right\rangle(Q)\,dv_g(R)$$
$$+2\int_M u(R)\left(\left\langle\nabla H(Q,R),\nabla\left(\Delta\eta_{r_o}(S,R)\right) + \Delta\left(\nabla\eta_{r_o}(S,R)\right)\right\rangle\right)dv_g(R)$$
$$+2\int_M u(R)\,H(Q,R)\left(\nabla^\mu\Delta\left(\nabla_\mu\eta_{r_o}(S,R)\right) - \Delta^2\eta_{r_o}(S,R)\right)dv_g(R)$$
$$-2\int_M u(R)\left(\Delta H(Q,R)\,\Delta\eta_{r_o}(S,R) + 2\left\langle\nabla^2\eta_{r_o}(S,R),\nabla^2 H(Q,R)\right\rangle\right)$$
$$-4\int_M u(R)\left\langle\nabla H(Q,R),\Delta\left(\nabla\eta_{r_o}(S,R)\right)\right\rangle dv_g(R)$$
$$+\int_M u(R)\left\langle\nabla\eta_{r_o}(S,R),\nabla\left(\Delta H(Q,R)\right) + \Delta\left(\nabla H(R,Q)\right\rangle dv_g(R)$$
$$+\frac{1}{V(M)}\int_M \eta_{r_o}(S,R)\,u(R)dv_g(R).$$

So by letting $Q = S$, we get the desired formula. □

Theorem 4. *Let $f \in K^n(U)$ and $u \in H^2_{2,loc}(U)$ a weak solution of the equation*

$$P(u) + fu = 0. \tag{3.11}$$

If $fu \in L^1_{loc}(U)$, then u is locally bounded on U.

PROOF. Let $I = [0,1]$ and denote by χ_I the characteristic function of I. Let also $0 < \delta \le \delta_o \le \frac{r_o}{4}$ where r_o is chosen so that $B(Q_o, 2r_o) \subset U$ with $Q_o \in U$.

First, we have

$$\left|\nabla^i_T\eta_\delta(Q,T)\right| \le c_i(n)\,\delta^{-i}\chi_I\left(\delta^{-1}d(T,Q)\right),\quad i = 1,....4.$$

On the other hand if we denote, respectively, by J_i, $i = 1,...,10$, the terms of the second right hand of the equality (3.10), we obtain

$$|J_1| \le \overline{c}_1(n)\int_M \frac{|(fu)(R)|\,\eta_\delta(Q,R)}{d(Q,R)^l}dv_g(R) \tag{3.12}$$
$$\le \overline{c}_1(n)\,\bar{j_1}$$

with

$$\bar{J_1} = \int_M \frac{|(fu)(R)|\,\eta_\delta(Q,R)}{d(Q,R)^l}dv_g(R)$$

where $l=(j+1)(n-4)-jn\left(1-\frac{1}{p}\right)$, $j\geq 1$, any integer and $\overline{c}_1(l)$ is a constant depending on l.

Letting $S\in B_{r_o}(Q_o)$, multiplying $\bar{j_1}$ by $\frac{|f(Q)|\eta_\delta(S,Q)}{d(S,Q)^l}$ and integrating over M, we get by the Fubini's formula

$$\overline{J}_1=\int_M\frac{|f(Q)|\,\eta_\delta(S,Q)}{d(S,Q)^l}\left(\int_M\frac{|(fu)(R)|\,\eta_\delta(Q,R)}{d(Q,R)^l}dv_g(R)\right)dv_g(Q)$$

$$=\int_M|(fu)(R)|\left(\int_M\frac{|f(Q)|\,\eta_\delta(S,Q)\,\eta_\delta(Q,R)}{d(S,Q)^l\,d(Q,R)^l}dv_g(Q)\right)dv_g(R)$$

and taking account of Lemma 7, we get

$$\left|\overline{J}_1\right|\leq c_1(n)\varphi_f(\delta,B_{3\delta}(S))\int_M|(fu)(R)|\,d(S,R)^{-l}\,\eta_{4\delta}(S,R)\,dv_g(R)$$

$$\leq c_1(n)\varphi_f(\delta,B_{3\delta}(S))\int_M|(fu)(R)|\,d(S,R)^{-l}\left(\eta_{4\delta}(S,R)-\eta_\delta(S,R)\right)dv_g(R)$$

$$+c_1(n)\varphi_f(\delta,B_{3\delta}(S))\int_M|(fu)(R)|\,d(S,R)^{-l}\,\eta_\delta(S,R)\,dv_R(R)$$

taking account of

$$\eta_{4\delta}(S,R)-\eta_\delta(S,R)=0\text{ , for }d(S,R)\leq\frac{\delta}{2}\text{ or }d(S,R)\geq 4\delta$$

we obtain

$$\left|\bar{J}_1\right|\leq 2c_1(n)\varphi_f(\delta,B_{3\delta}(S))\left(\frac{\delta}{2}\right)^{-l}\|fu\|_{L^1(B_{4\delta+r_o}(Q_o))}+c_1(n)\varphi_f(\delta,B_{3\delta}(S))\,\overline{J}_1$$

and since $\varphi_f(\delta,B_{3\delta}(S))\to 0^+$ as $\delta\to 0^+$, we choose $\delta>0$ such that

$$1-c_1(n)\varphi_f(\delta,B_{3\delta}(S))>0.$$

Hence

$$\left|\overline{J}_1\right|<+\infty$$

and

$$|J_1|<+\infty$$

By the Giraud's Lemma [**14**], we get

$$|J_2|\leq C_1\int_M\frac{\left|u(R)\left\{\Delta^2\eta_\delta(Q,R)\right\}\right|}{d(Q,R)^l}dv_g(R)$$

$$+C_2\int_M\frac{|(ua)(R)|\,|\nabla_R\eta_\delta(Q,R)|}{d(Q,R)^{l-1}}dv_g(R)+C_o\int_M\frac{|u(R)|\,|\nabla a(R)|\,\left|\nabla\eta_{r_o}(S,R)\right|}{d(Q,R)^{l-2}}dv_g(R)$$

and by Lemma 7, we obtain

$$|J_2|\leq c_2(n)\,\delta^{-4}\left(\frac{\delta}{2}\right)^{-l}\|u\|_{L^1(B_{2\delta+r_o})}+c_1(n)\,\delta^{-1}\left(\frac{\delta}{2}\right)^{-l-1}\|a\|_p\,\|u\|_{L^{\frac{p}{p-1}}(B_{2\delta+r_o})}$$

$$+c_o(n)\delta^{-1}\left(\frac{\delta}{2}\right)^{-l}\|a\|_p\,\|\nabla u\|_{L^{\frac{p}{p-1}}(B_{2\delta+r_o})}.$$

Also

$$|J_3|\leq\left(2^3c_2(n)\left(\frac{\delta}{2}\right)^{-l}+2^5c_3(n)\left(\frac{\delta}{2}\right)^{-l-2}+2^5c_4(n)\left(\frac{\delta}{2}\right)^{-l-4}\right)\|u\|_{L^1(B_{2\delta+r_o})}.$$

By the same procedure as above and applying repeatedly Lemma 7, we get that all the remaining terms of the formula (3.10) are bounded and the solution u of the equation 3.11 is locally bounded. □

4. Q-curvature type equation

Let (M,g) be a compact n-dimensional Riemannian manifold, $n \geq 5$, we consider the following fourth order equation

$$\Delta^2 u - \nabla^i \left(a(x)\nabla_i u\right) + b(x)u = f\,|u|^{N-2}\,u \tag{4.1}$$

where $a \in L^s(M)$, $b \in L^p(M)$, with $s > \frac{n}{2}$, $p > \frac{n}{4}$, $f \in C^\infty(M)$ a positive function and $N = \frac{2n}{n-4}$. To solve the equation (4.1), we use the variational method.

For any $u \in H_2(M)$, we let

$$J(u) = \int_M (\Delta u)^2\, dv_g + \int_M a(x)\,|\nabla u|^2\, dv_g + \int_M b(x)u^2 dv_g$$

be the energy functional and consider the Sobolev quotient, for any $u \in H_2(M) - \{0\}$

$$Q(u) = \frac{J(u)}{\left(\int f\,|u|^N\, dv_g\right)^{\frac{2}{N}}}.$$

Let

$$A = \left\{u \in H_2(M) : \int_M f\,|u|^N\, dv_g = 1\right\}$$

obviously $A \neq \phi$.

Put

$$Q(M) = \inf_{u \in H_2(M)-\{0\}} Q(u) = \inf_{u\in A} J(u).$$

Theorem 5. *Suppose that $Q(M) < \sup_{x\in M} f(x)K(n,2)$. The equation (4.1) has a non trivial solution u satisfying $J(u) = Q(M)$ and $u \in A$. Moreover $u \in C^{3-\frac{p}{n}+E\left(\frac{p}{n}\right),\beta}(M)$ with $\beta \in \left]0, 1-\frac{p}{n}+E\left(\frac{p}{n}\right)\right[$.*

Proof. First we show that $Q(M)$ is finite. For any $u \in A$,

$$J(u) \geq \|\Delta u\|_2^2 - \|a\|_s\,\|\nabla u\|_{2s'}^2 - \|b\|_p\,\|u\|_N^p\,V(M)^{1-\frac{p}{N}} \tag{4.2}$$

with

$$s' = \frac{s}{s-1}.$$

Since $s > \frac{n}{2}$, Sobolev inequality leads to

$$\|\nabla u\|_{2s'}^2 \leq K(2,n)^2\,\|\nabla\,|\nabla u|\|_2^2 + A_1\,\|\nabla u\|_2^2$$

and since

$$|\nabla\,|\nabla u|| \leq \left|\nabla^2 u\right|$$

we obtain

$$\|\nabla u\|_{2s'}^2 \leq K(2,n)^2\,\left\|\nabla^2 u\right\|_2^2 + A_1\,\|\nabla u\|_2^2. \tag{4.3}$$

Now by an interpolation inequality (see Aubin [1], p. 93), for any $\eta > 0$ there is a constant $C(\eta) > 0$ such that for any $u \in C^\infty(M)$

$$\|\nabla u\|_2^2 \leq \eta\,\left\|\nabla^2 u\right\|_2^2 + C\left(\eta\right)\|u\|_2^2. \tag{4.4}$$

Independently, we have on compact manifolds we have the well known inequality (see Aubin [1], p.113))

$$\left\|\nabla^2 u\right\|_2^2 = \|\Delta u\|_2^2 - \int_M R_{ij}\nabla^i u \nabla^j u dv_g$$

so there is a constant $\beta > 0$ such that

$$\left\|\nabla^2 u\right\|_2^2 \leq \|\Delta u\|_2^2 + \beta \|\nabla u\|_2^2 . \tag{4.5}$$

Combining (4.4), (4.5), we get

$$(1-\eta\beta)\|\nabla u\|_2^2 \leq \eta \|\Delta u\|_2^2 + C(\eta)\|u\|_2^2 . \tag{4.6}$$

Choosing η so that $1-\eta\beta > 0$ and taking account of the density of $C^\infty(M)$ in H_2 and the inequality (4.3), we obtain

$$\|\nabla u\|_{2s'}^2 \leq \left(\left(K(2,n)^2 + A_1\right)\frac{\eta}{1-\eta\beta}\right)\|\Delta u\|_2^2 + \left(K(2,n)^2 + A_1\right)\frac{C(\eta)}{1-\eta\beta}\|u\|_2^2$$

$$\leq \left(K(2,n)^2 + O(\eta)\right)\|\Delta u\|_2^2 + \left(K(2,n)^2 + A_1\right)\frac{C(\eta)}{1-\eta\beta}\|u\|_N^2 V(M)^{1-\frac{2}{N}}. \tag{4.7}$$

Consequently, (4.2) becomes

$$J(u) \geq \left(1 - \|a\|_S \left(K(2,n)^2 + O(\eta)\right)\right)\|\Delta u\|_2^2 \tag{4.8}$$

$$- \|a\|_s \left(K(2,n)^2 + A_1\right)\frac{C(\eta)}{1-\eta\beta}\|u\|_N^2 V(M)^{1-\frac{2}{N}} - \|b\|_p \|u\|_N^p V(M)^{1-\frac{p}{N}}.$$

Since $u \in A$, we can write that

$$\|u\|_N \leq \left(\min_{x\in M} f(x)\right)^{-\frac{1}{N}} \tag{4.9}$$

and η being arbitrary, so if

$$\|a\|_s \leq K(2,n)^{-2}$$

then

$$J(u) \geq -\|a\|_s \left(K(2,n)^2 + A_1\right)\frac{C(\eta)}{1-\eta\beta}\left(\min_{x\in M} f(x)\right)^{-\frac{2}{N}} V(M)^{1-\frac{2}{N}} \tag{4.10}$$

$$- \|b\|_p \left(\min_{x\in M} f(x)\right)^{-\frac{p}{N}} V(M)^{1-\frac{p}{N}} > -\infty.$$

Let $(u_i)_i \subset A$ be a minimizing sequence of the functional J i.e.

$$J(u_i) = Q(M) + o(1).$$

By (4.8) and (4.10), we obtain

$$\|\Delta u_i\|_2 < +\infty$$

and by (4.6), (4.9) and Hölder's inequality we infer that

$$\|u_i\|_{H_2(M)} < +\infty.$$

Up to a subsequence, there is $u \in H_2(M)$ such that

- $u_i \to u$ weakly in $H_2(M)$

· $\nabla u_i \to \nabla u$ strongly in $L^s(M)$, $s < 2^* = \frac{2n}{n-2}$
· $u_i \to u$ strongly in $L^r(M)$, $r < N$
· $u_i \to u$ a.e. in M.

Letting $v_i = u_i - u$, we conclude that
$\int_M \Delta u \Delta v_i dv_g \to 0$, $\int_M a g(\nabla u, \nabla v_i) dv_g \to 0$ as $i \to +\infty$.
and

$$\int_M |buv_i|\, dv_g \le \|b\|_p \left(\int_M |u|^{\frac{p}{p-1}} |v_i|^{\frac{p}{p-1}} dv_g \right)^{1-\frac{1}{p}} \le \|b\|_p \|u\|_{\frac{2p}{p-1}} \|v_i\|_{\frac{2p}{p-1}}$$

i.e. $\int_M buv_i dv_g \to 0$, since $\frac{2p}{p-1} < N$.

Consequently

$$\begin{aligned} J(u_i) &= J(u) + J(v_i) + 2\int_M \Delta u \Delta v_i dv_g + 2\int_M a g(\nabla u, \nabla v_i) dv_g + 2\int_M buv_i dv_g \\ &= J(u) + J(v_i) + o(1) \\ &= J(u) + \|\Delta v_i\|_2^2 + o(1). \end{aligned}$$

By definition of $Q(M)$, $J(u) \ge Q_M \left(\int_M f |u|^N dv_g \right)^{\frac{2}{N}}$ and $J(u_i) = Q(M) + o(1)$ and by definition of the sequence (u_i), we obtain

$$Q_M \left(\int_M f |u|^N dv_g \right)^{\frac{2}{N}} + \|\Delta v_i\|_2^2 \le Q(M) + o(1). \tag{4.11}$$

Brezis -Lieb lemma allows us to write

$$1 = \int_M f |u_i|^N dv_g = \int_M f |u|^N dv_g + \int_M f |v_i|^N dv_g + o(1)$$

hence

$$1 \le \left(\int_M f |u|^N dv_g \right)^{\frac{2}{N}} + \left(\int_M f |v_i|^N dv_g \right)^{\frac{2}{N}} + o(1)$$

and the inequality (4.11) will be written as

$$\|\Delta v_i\|_2^2 \le Q(M) \left(\int_M f |v_i|^N dv_g \right)^{\frac{2}{N}} + o(1). \tag{4.12}$$

By Sobolev's inequality we infer that

$$\|\Delta v_i\|_2^2 \le Q(M) \left(\sup_{x\in M} f(x) \right)^{\frac{2}{N}} \left(K(n,2)^2 + \varepsilon \right) \|\Delta v_i\|_2^2 + o(1).$$

Finally

$$\left(1 - Q(M) \left(\sup_{x\in M} f(x) \right)^{\frac{2}{N}} \left(K(n,2)^2 + \varepsilon \right) \right) \|\Delta v_i\|_2^2 \le o(1).$$

If we let

$$Q(M) < \left(\sup_{x\in M} f(x) \right)^{-\frac{2}{N}} K(n,2)^{-2}$$

and choosing $\varepsilon > 0$ small enough such that

$$1 - Q(M) \left(\sup_{x\in M} f(x) \right)^{\frac{2}{N}} \left(K(n,2)^2 + \varepsilon \right) > 0$$

we obtain

$$\|\Delta v_i\|_2^2 = o(1).$$

Hence (v_i) converges strongly to 0 in $H_2(M)$ and (u_i) converges strongly to u in $H_2(M)$ and in $L^N(M)$. We conclude that $u \in A$ is a non trivial solution of the equation

$$\Delta^2 u - \nabla^{\mu}(a\nabla_{\mu} u) + bu = f\,|u|^{N-2}\,u. \tag{4.13}$$

Now, we are going to establish the regularity of u to do so we first show that the function $h = -f\,|u|^{N-2}$ is a Kato- Stummel function. Using Hölder's inequality, we get

$$\begin{aligned}\sup_{Q\in M}\int_{B_t(Q)} \frac{|h(S)|\,\chi_U\,(S)}{d(Q,S)^l}dv_g(S) &\leq \int_M \frac{|f(S)|\,|u(S)|^{N-2}}{d(Q,S)^l}dv_g(S)\\ &\leq \max_{S\in M}|f(S)|\,\|u\|_{N-2,\rho^{-l}}^{N-2}.\end{aligned}$$

By Lemma 3, we get that

$$\|u\|_{N-2,\rho^{-l}}^{N-2} \leq C\left(\|\Delta u\|_2^2 + \|u\|_2^2\right)$$

where $C > 0$ is some constant.

The remaining part to check that the function h is a Kato-Stummel is the same as in [**17**] so we omit it.

We conclude that the solution u of the equation (4.13) is locally bounded and in fact bounded since the manifold M is compact.

Hence, we get that

$$\Delta^2 u \in L^p(M).$$

By classical regularity theorem and the Sobolev embedding $\Delta u \in H_2^p(M) \subset C^{1-\left(\frac{p}{n}-E(\frac{p}{n})\right),\beta}\ (M)$ with $\beta \in \left]0, 1-\left(\frac{p}{n}-E(\frac{p}{n})\right)\right[$. Consequently $u \in C^{3-\left(\frac{p}{n}-E(\frac{p}{n})\right),\beta}\ (M)$. □

Let α,γ be real number and consider the equation

$$\Delta^2 u - \nabla^{\mu}\left(\frac{a}{\rho^{\gamma}}\nabla_{\mu} u\right) + \frac{Q_g u}{\rho^{\alpha}} = f\,|u|^{N-2}\,u \tag{4.14}$$

where a and Q_g are smooth functions on M.

We put, for any $u \in H_2(M)$

$$J_{\gamma,\alpha}(u) = \|\Delta u\|_2^2 + \int_M \frac{a}{\rho^{\gamma}}\,|\nabla u|^2\,dv_g + \int_M \frac{Q_g u^2}{\rho^{\alpha}}dv_g$$

$$Q_{\gamma,\alpha}(u) = \frac{J_{\gamma,\alpha}(u)}{\left(\int_M f\,|u|^N\,dv_g\right)^{\frac{2}{N}}}$$

and

$$Q_{\gamma,\alpha}(M) = \inf_{u\in H_2(M)-\{0\}} Q_{\gamma,\alpha}(u) = \inf_{u\in A} J_{\alpha}(u)$$

where $A = \left\{u \in H_2(M) : \int_M f\,|u|^N\,dv_g = 1\right\}$.

As a corollary to Theorem 5, we have

Theorem 6. *Let $\gamma \in (0,2)$ $\alpha \in (0,4)$, if $Q_{\gamma,\alpha}(M) < (\sup f(x))^{-\frac{N}{2}}\,K(n,2)^{-2}$, then the equation (4.14) has a non trivial solution $u_{\gamma,\alpha} \in A$, which fulfilled $J_{\gamma,\alpha}(u) = Q_{\gamma,\alpha}(M)$ and $u_{\gamma,\alpha} \in C^{3-\left(\frac{p}{n}-E(\frac{p}{n})\right),\beta}$ with $\beta \in \left]0, 1-\left(\frac{p}{n}-E\left(\frac{p}{n}\right)\right)\right[$.*

PROOF. Let $\overline{a} = \frac{a}{\rho^\gamma}$, $b = \frac{Q_g}{\rho^\alpha}$, if $\gamma \in (0,2)$, $\alpha \in (0,4)$ then by Lemma 2, $\overline{a} \in L^s(M), b \in L^p(M)$ with $0 < \gamma < 2$, $0 < \alpha < \frac{n}{p} < 4$ and Theorem 6 follows from Theorem 5. □

5. The sharp case $\gamma = 2$, $\alpha = 4$

In the previous section we have shown that if $\gamma \in (0,2)$, $\alpha \in (0,4)$ and $Q_{\gamma,\alpha}(M) < (\sup f(x))^{-\frac{N}{2}} K(n,2)^{-2}$, the equation

$$\Delta^2 u - \nabla^\mu(\frac{a}{\rho^\gamma}\nabla_\mu u) + \frac{Q_g u}{\rho^\alpha} = f\,|u|^{N-2}\,u \tag{5.1}$$

has a solution $u_\alpha \in A = \left\{u \in H_2(M) : \int_M f\,|u|^N\,dv_g = 1\right\}$ such that $J_{\gamma,\alpha}(u_\alpha) = Q_{\gamma,\alpha}(M)$.

We will show the following

Theorem 7. *Suppose that* $1 + Q_g(P)K(n,2,-4)^2 > 0$ *and* $Q_{2,4}(M)(K(n,2)^2\,\|f\|_\infty^{\frac{2}{N}} < 1 + Q_g(P)K(n,2,-4)^2$, *then the equation*

$$\Delta^2 u - \nabla^\mu(\frac{a}{\rho^2}\nabla_\mu u) + \frac{Q_g u}{\rho^4} = f\,|u|^{N-2}\,u$$

has a non trivial solution $u_{2,4} \in A$, *which fulfilled* $J_{2,4}(u) = Q_4(M)$.

PROOF. First we show that $Q_{2,4}(M)$ is finite. Since Q_g is continuous, for any $\varepsilon > 0$, there is $\delta > 0$ such that for any $Q \in M$, with $d(P,Q) < \delta$ and $Q_g(Q) > Q_g(P) - \varepsilon$, then

$$\int_M \frac{Q_g u^2}{\rho^4}dv_g \geq (Q_g(P)-\varepsilon)\int_{B_\delta(P)} \frac{u^2}{\rho^4}dv_g - \frac{\|Q_g\|_\infty}{\delta^4}\int_{M-B_\delta(P)} u^2 dv_g. \tag{5.2}$$

By Lemma (3), we have

$$\int_{B_\delta(P)} \frac{u^2}{\rho^4}dv_g \leq \left(K(n,2,-4)^2+\varepsilon\right)\|\Delta u\|_2^2 + A(\varepsilon)\,\|u\|_2^2\,. \tag{5.3}$$

Combining (5.2) and (5.3), we get

$$\begin{aligned}\int_M \frac{Q_g u^2}{\rho^4}dv_g &\geq (\min(Q_g(P),0)-\varepsilon)\left(K(n,2,-4)^2+\varepsilon\right)\|\Delta u\|_2^2 \\ &+ \left((\min(Q_g(P),0)-\varepsilon)\,A(\varepsilon) - \frac{\|Q_g\|_\infty}{\delta^4}\right)\|u\|_2^2\end{aligned} \tag{5.4}$$

and since we have also

$$\|u\|_2^2 \leq \left(\min_{x\in M} f(x)\right)^{-\frac{p}{N}} V(M)^{1-\frac{p}{N}}$$

the inequality

$$\begin{aligned}\int_M \frac{Q_g u^2}{\rho^4}dv_g &\geq (\min(Q_g(P),0)-\varepsilon)\left(K(n,2,-4)^2+\varepsilon\right)\|\Delta u\|_2^2 \\ &+ \left((\min(Q_g(P),0)-\varepsilon)\,A(\varepsilon) - \frac{\|Q_g\|_\infty}{\delta^4}\right)\left(\min_{x\in M} f(x)\right)^{-\frac{p}{N}} V(M)^{1-\frac{p}{N}}.\end{aligned} \tag{5.5}$$

Also, we have

$$\int_M \frac{S_g |\nabla u|^2}{\rho^2} dv_g \geq (\min(S_g(P),0)-\varepsilon) \int_{B_\delta(P)} \frac{|\nabla u|^2}{\rho^2} dv_g - \frac{\|S_g\|_\infty}{\delta^2} \int_{M-B_\delta(P)} |\nabla u|^2 dv_g$$

and by Hardy's inequality (2)

$$\int_M \frac{S_g |\nabla u|^2}{\rho^2} dv_g \geq (\min(S_g(P),0)-\varepsilon) K_1(2,n,-2)+\varepsilon) \|\Delta u\|_2^2$$
$$+\left((\min(S_g(P),0)-\varepsilon) A_1(\varepsilon) - \frac{\|S_g\|_\infty}{\delta^2}\right) \|\nabla u\|_2^2$$

and taking again of relation (4.7)

$$\int_M \frac{S_g |\nabla u|^2}{\rho^2} dv_g \geq$$
$$\left((\min(S_g(P),0)-\varepsilon) K_1(2,n,-2)+\varepsilon) + \left((\min(S_g(P),0)-\varepsilon) A_1(\varepsilon) - \frac{\|S_g\|_\infty}{\delta^2}\right) \frac{\eta}{1-\eta\beta}\right) \|\Delta u\|_2^2$$
$$+\left((\min(S_g(P),0)-\varepsilon) A_1(\varepsilon) - \frac{\|S_g\|_\infty}{\delta^2}\right) \frac{C(\eta)}{1-\eta\beta} \|u\|_2^2$$

As in previous sections, we get

$$J_4(u) \geq [1+(\min(S_g(P),0)-\varepsilon) K_1(2,n,-2)+\varepsilon)$$
$$+\left((\min(S_g(P),0)-\varepsilon) A_1(\varepsilon) - \frac{\|S_g\|_\infty}{\delta^2}\right) \frac{\eta}{1-\eta\beta} + (\min(Q_g(P,0)-\varepsilon) \left(K_2(n,2,-4)^2+\varepsilon\right)\Big] \|\Delta u\|_2^2$$
$$+\left((\min(S_g(P),0)-\varepsilon) A_1(\varepsilon) - \frac{\|S_g\|_\infty}{\delta^2}\right) \frac{C(\eta)}{1-\eta\beta} \|u\|_N^2 V(M)^{1-\frac{2}{N}}$$
$$+\left((\min(Q_g(P,0)-\varepsilon) A(\varepsilon) - \frac{\|Q_g\|_\infty}{\delta^4}\right) \left(\min_{x\in M} f(x)\right)^{-\frac{p}{N}} V(M)^{1-\frac{p}{N}}. \tag{5.6}$$

If

$$1+Q_g(P)K(n,2,-4)>0$$

and letting ε and η small enough the inequality (5.6) becomes

$$J_4(u) \geq \left((\min(S_g(P),0)-\varepsilon) A_1(\varepsilon) - \frac{\|S_g\|_\infty}{\delta^2}\right) \frac{C(\eta)}{1-\eta\beta} \left(\min_{x\in M} f(x)\right)^{-\frac{2}{N}} V(M)^{1-\frac{2}{N}}$$
$$+\left((\min(Q_g(P,0)-\varepsilon) A(\varepsilon) - \frac{\|Q_g\|_\infty}{\delta^4}\right) \left(\min_{x\in M} f(x)\right)^{-\frac{p}{N}} V(M)^{1-\frac{p}{N}} > -\infty.$$

Consequently

$$Q_4(M) > -\infty.$$

In the second step, we will show that $Q_{\gamma,\alpha}(M) \to Q_{2,4}(M)$ as $\alpha \to 4$.

Let $0<\delta<\min(1,\delta(M))$, where $\delta(M)$ denotes the injectivity radius, then

$$\int_M \frac{Q_g u^2}{\rho^\alpha} dv_g = \int_{B_\delta(P)} \frac{Q_g u^2}{\rho^\alpha} dv_g + \int_{M-B_\delta(P)} \frac{Q_g u^2}{\rho^\alpha} dv_g$$

and by the Lebesgue dominated convergence theorem, we obtain that

$$\int_M \frac{Q_g u^2}{\rho^\alpha} dv_g \to \int_M \frac{Q_g u^2}{\rho^4} dv_g \text{ as } \alpha \to 4^-$$

The same arguments are also true for

$$\int_M \frac{S_g |\nabla u|^2}{\rho^\gamma} dv_g \to \int_M \frac{S_g |\nabla u|^2}{\rho^2} dv_g \text{ as } \gamma \to 2^-$$

Hence

$$J_{\gamma,\alpha}(u) \to J_{2,4}(u) \text{ as } \gamma \to 2^- \text{ and } \alpha \to 4^-$$

and by passing to the infimum over u such that $\int_M f |u|^N dv_g = 1$, we get

$$Q_{\gamma,\alpha}(M) \to Q_{2,4}(M) \text{ as } \gamma \to 2^- \text{ and } \alpha \to 4^-.$$

In this third step, we will show that the sequence (u_α) is uniformly bounded in $H_2(M)$. This sequence satisfies

$$\|u_\alpha\|_2^2 \le \left(\min_{x\in M} f(x)\right)^{-\frac{2}{N}} V(M)^{1-\frac{2}{N}}$$

and if $1 + Q_g(P)K(n,2,4) > 0$, then

$$\|\Delta u_\alpha\|_2^2 < +\infty$$

and by the inequality (4.7),

$$\|\nabla u_\alpha\|_2^2 < +\infty.$$

Up to a subsequence (u_α) converges weakly in $H_2(M)$, $L^N(M)$, $L^2(M,\rho^{-4})$ and strongly to u in $L^r(M)$ with $r < \frac{2n}{n-4}$ and ∇u_α converges strongly to ∇u in $L^s(M)$ with $s < \frac{2n}{n-2}$. For any $v \in H_2(M)$

(5.7)
$$\int_M \Delta u_\alpha \Delta v dv_g + \int_M \frac{S_g}{\rho^\gamma} g(\nabla u_\alpha, \nabla v) dv_g + \int_M \frac{Q_g}{\rho^\alpha} u_\alpha v dv_g = Q_\alpha(M) \int_M f |u_\alpha|^{N-2} u_\alpha v dv_g.$$

The weak convergence in $H_2(M)$ and the strong convergence of $\nabla u_\alpha \to \nabla u$ allow us to write

$$\int_M \Delta u_\alpha \Delta v dv_g \to \int_M \Delta u \Delta v dv_g$$

and

$$\int_M \frac{S_g}{\rho^\gamma} g(\nabla u_\alpha, \nabla v) dv_g \to \int_M \frac{S_g}{\rho^2} g(\nabla u, \nabla v) dv_g$$

The convergence of the third integral

$$\left| \int_M \frac{Q_g}{\rho^\alpha} u_\alpha v dv_g - \int_M \frac{Q_g}{\rho^4} u v dv_g \right| \le \left| \int_M \frac{Q_g}{\rho^4} v \left(u_\alpha - u\right) dv_g \right| + \left| \int_M \frac{Q_g}{\rho^4} u_\alpha v dv_g - \int_M \frac{Q_g}{\rho^\alpha} u_\alpha v dv_g \right|$$

is assured by the weak convergence in $L^2(M,\rho^{-4})$ and the dominated Lebesque convergence theorem. Since (u_α) is bounded in $L^N(M)$, the sequence $\left(|u_\alpha|^{N-2} u_\alpha\right)$ is bounded in $L^{\frac{N}{N-1}}(M)$, hence $Q_{\gamma,\alpha}(M) \int_M f |u_\alpha|^{N-2} u_\alpha v dv_g \to Q_{2,4}(M) \int_M f |u|^{N-2} u v dv_g$. Consequently u is a weak solution of equation 5.1.

In this last step, we will prove that u is not trivial. By Sobolev's inequality (5), we have

$$\|f\|^{-\frac{2}{N}} \le \|u_\alpha\|_N^2 \le (K(n,2)^2 + \varepsilon) \|\Delta u_\alpha\|_2^2 + A(\varepsilon) \|u_\alpha\|_2^2 . \tag{5.8}$$

Since u_α are solutions (5.1), for any $\overline{\eta}>0$,

(5.9)
$$\|\Delta u_\alpha\|_2^2=(1+\overline{\eta})\left(Q_{\gamma,\alpha}(M)-\int_M\frac{S_g}{\rho^\gamma}\left|\nabla u_\alpha\right|^{\,2}dv_g-\int_M\frac{Q_gu_\alpha^2}{\rho^\alpha}dv_g\right)-\overline{\eta}\,\|\Delta u_\alpha\|_2^2\,.$$

And since as it is shown in previous sections

$$\|\nabla u_\alpha\|_2^2\leq\frac{\eta}{1-\eta\beta}\|\Delta u_\alpha\|_2^2+\frac{C(\eta)}{1-\eta\beta}\|u_\alpha\|_2^2 \tag{5.10}$$

where $\beta>0$ and arbitrary $\eta>0$ such that $1-\eta\beta>0$.

And also by Sobolev's inequality (5)

$$\int_M\frac{Q_gu_\alpha^2}{\rho^\alpha}dv_g\geq(Q_g(P)-\varepsilon)\int_{B_\delta(P)}\frac{Q_gu_\alpha^2}{\rho^\alpha}dv_g-\frac{\|a\|_\infty}{\delta^4}\|u_\alpha\|_2^2$$

(5.11)
$$\geq(\min(Q_g(P),0)-\varepsilon)\left[(K(n,2,-\alpha)^2+\varepsilon_1)\|\Delta u_\alpha\|_2^2+A(\varepsilon_1)\|u_\alpha\|_2^2\right]-\frac{\|a\|_\infty}{\delta^4}\|u_\alpha\|_2^2\,.$$

Plugging (5.10) and (5.11) in (5.9), we get

$$\|\Delta u_\alpha\|_2^2\leq(1+\overline{\eta})\left[Q_{\gamma,\alpha}(M)+\frac{\eta}{1-\eta\beta}\|a\|_\infty\|\Delta u_\alpha\|_2^2+\frac{C(\eta)}{1-\eta\beta}\|u_\alpha\|_2^2\right.$$

(5.12)
$$\left.-(\min(Q_g(P),0)-\varepsilon)\left[(K(n,2,-\alpha)^2+\varepsilon_1)\|\Delta u_\alpha\|_2^2+A(\varepsilon_1)\|u_\alpha\|_2^2\right]+\frac{\|a\|_\infty}{\delta^4}\|u_\alpha\|_2^2\right]$$
$$-\overline{\eta}\,\|\Delta u_\alpha\|_2^2$$

and taking $\overline{\eta}$ so that

$$\overline{\eta}=(1+\overline{\eta})\frac{\eta}{1-\eta\beta}\|a\|_\infty$$

we get

$$\left[1+(1+\overline{\eta})\left(\min(Q_g(P),0)-\varepsilon\right)\left(K(n,2,-\alpha)^2+\varepsilon_1\right)\right]\|\Delta u_\alpha\|_2^2\leq$$

(5.13)
$$(1+\overline{\eta})\left[Q_{\gamma,\alpha}(M)+\left(\frac{C(\eta)}{1-\eta\beta}-(\min(Q_g(P),0)-\varepsilon)\,A(\varepsilon_1)+\frac{\|a\|_\infty}{\delta^4}\right)\|u_\alpha\|_2^2\right].$$

So if

$$1+Q_g(P)K(n,2,-\alpha)^2>0 \tag{5.14}$$

by letting $\varepsilon,\varepsilon_1$, and η small enough, we get

(5.15)
$$\|\Delta u_\alpha\|_2^2\leq\frac{(1+\overline{\eta})Q_\alpha(M)+\left(\frac{C(\eta)}{1-\eta\beta}-(\min(Q_g(P),0)-\varepsilon)\,A(\varepsilon_1)+\frac{\|a\|_\infty}{\delta^4}\right)\|u_\alpha\|_2^2}{1+(1+\overline{\eta})\left(\min(Q_g(P),0)-\varepsilon\right)\left(K(n,2,-\alpha)^2+\varepsilon_1\right)}$$

and replacing in (5.8)

$$\left[(K(n,2)^2+\varepsilon)\frac{\left(\frac{C(\eta)}{1-\eta\beta}-(\min(Q_g(P),0)-\varepsilon)\,A(\varepsilon_1)+\frac{\|a\|_\infty}{\delta^4}\right)}{1+(1+\overline{\eta})\left(\min(Q_g(P),0)-\varepsilon\right)\left(K(n,2,-\alpha)^2+\varepsilon_1\right)}+A(\varepsilon)\right]\|u_\alpha\|_2^2\geq$$
$$\|f\|_\infty^{-\frac{2}{N}}-\frac{(1+\overline{\eta})Q_{\gamma,\alpha}(M)(K(n,2)^2+\varepsilon)}{1+(1+\overline{\eta})\left(\min(Q_g(P),0)-\varepsilon\right)\left(K(n,2,-\alpha)^2+\varepsilon_1\right)}.$$

Since

$$Q_\alpha(M) = Q_4(M) + o(1)$$

and in addition of (5.14) and the following assumption

$$Q_{2,4}(M)(K(n,2)^2 \|f\|_\infty^{\frac{2}{N}} < 1 + Q_g(P)K(n,2,-4)^2$$

we get that the solution u of the sharp equation is not trivial. $\square$

6. Geometric interpretation

Consider a flat manifold (M,h) for example a flat torus and let $g = Ah$ where $A = e^{-\rho^{2-\rho}}$.

The respective expressions of the Ricci tensor and the scalar curvature of (M,g) are then

$$Ric_g = -\frac{A}{2}\left[-\Delta \log A + \frac{n}{2}|\nabla \log A|^2\right]\delta_{ij}.$$

and that of the scalar curvature is

$$R_g = -\frac{n}{2}\left[-\Delta \log A + \frac{n}{2}|\nabla \log A|^2\right]$$

We infer the following expressions

$$\frac{(n-2)^2+4}{2(n-1)(n-2)}R_g.g - \frac{4}{n-2}Ric_g = -\frac{(n+2)(n-2)^2+4}{4(n-1)(n-2)}\left[-\Delta \log A + \frac{n}{2}|\nabla \log A|^2\right].g.$$

Put

$$\widetilde{\alpha} = -\frac{(n+2)(n-2)^2+4}{4(n-1)(n-2)}\left[-\Delta \log A + \frac{n}{2}|\nabla \log A|^2\right]. \tag{6.1}$$

hand

$$\widetilde{\beta} = \frac{n-4}{2}Q_g^n$$

then

$$\begin{aligned}
\widetilde{\beta} &= \frac{n-4}{4(n-1)}\Delta_g R_g + \frac{\left[n^3-4n^2+16(n-1)\right](n-4)}{16(n-1)^2(n-2)^2}R_g^2 - \frac{n-4}{(n-2)^2}|Ric_g|^2 \\
&= -\frac{n(n-4)}{8(n-1)}\left[-\Delta^2 \log A + \frac{n}{2}\Delta|\nabla \log A|^2\right] \qquad (6.2)\\
&\quad + \frac{n\left(n^4-4n^3-16n^2+48n-32\right)(n-4)}{64(n-1)^2(n-2)^2}\left[-\Delta \log A + \frac{n}{2}|\nabla \log A|^2\right]^2.
\end{aligned}$$

We have

$$\begin{aligned}
\widetilde{\alpha}^2 - 4\widetilde{\beta} &= \left(\left(\frac{(n+2)(n-2)^2+4}{4(n-1)(n-2)}\right)^2 - \frac{n\left(n^4-4n^3-16n^2+48n-32\right)(n-4)}{16(n-1)^2(n-2)^2}\right)\times \\
&\quad \left[-\Delta \log A + \frac{n}{2}|\nabla \log A|^2\right]^2 + \frac{n(n-4)}{8(n-1)}\left(-\Delta^2 \log A + \frac{n}{2}\Delta|\nabla \log A|^2\right) \\
&= \frac{4\left(n^5-n^4-18n^3+48n^2-56n+36\right)}{16(n-1)^2(n-2)^2}\left(-\Delta^2 \log A + \frac{n}{2}|\nabla \log A|^2\right)^2 \\
&\quad + \frac{n(n-4)}{8(n-1)}\left(-\Delta^2 \log A + \frac{n}{2}\Delta|\nabla \log A|^2\right).
\end{aligned}$$

If we let

$$a_n = \frac{4\left(n^5 - n^4 - 18n^3 + 48n^2 - 56n + 36\right)}{16\left(n-1\right)^2\left(n-2\right)^2}$$

and

$$b_n = \frac{n\left(n-4\right)}{8(n-1)}$$

we get

(6.3)

$$\widetilde{\alpha}^2 - 4\widetilde{\beta} = \left(-\Delta^2 \log A + \frac{n}{2}\left|\nabla \log A\right|^2\right)\left[a_n\left(-\Delta^2 \log A + \frac{n}{2}\left|\nabla \log A\right|^2\right) + b_n\right]$$

with

$$A = e^{-\rho^{2-\alpha}}.$$

From the radial expression of the laplcian

$$\Delta = -\frac{1}{\rho^{n-1}}\frac{\partial}{\partial \rho}\left(\rho^{n-1}\frac{\partial}{\partial \rho}\right)$$

we get

$$\Delta^2 \log A = -\alpha\left(2-\alpha\right)\left(n-\alpha\right)\left(n-2-\alpha\right)\rho^{-\alpha-2}$$

and

$$\Delta\left|\nabla \log A\right|^2 = 2\left(2-\alpha\right)^2\left(\alpha-1\right)\left(n-2-\alpha\right)\rho^{-2\alpha}$$

hence

$$-\Delta^2 \log A + \frac{n}{2}\Delta\left|\nabla \log A\right|^2 = \tag{6.4}$$

$$\left(2-\alpha\right)\left[-\alpha\left(n-\alpha\right)\left(n-2-\alpha\right)\rho^{\alpha-2} + \left(2-\alpha\right)\left(\alpha-1\right)n\left(n-2\alpha\right)\right]\rho^{-2\alpha}.$$

Now if

$$1 < \alpha < 2$$

and

$$\rho < \rho_1 = \left(\frac{\alpha\left(n-\alpha\right)\left(n-2-\alpha\right)}{\left(2-\alpha\right)\left(\alpha-1\right)n\left(n-2\alpha\right)}\right)^{\frac{1}{2-\alpha}}$$

then

$$-\Delta^2 \log A + \frac{n}{2}\Delta\left|\nabla \log A\right|^2 < 0.$$

Consequently

$$\Delta\widetilde{\alpha} > 0. \tag{6.5}$$

Also $\widetilde{\alpha}$ will be positive if

$$\rho < \rho_2 = \left(\frac{2\left(n-\alpha\right)}{n\left(2-\alpha\right)}\right)^{\frac{1}{2-\alpha}}.$$

Finally $\widetilde{\alpha}^2 - 4\widetilde{\beta}$ will be positive if

$$\rho < \rho_3 = \left(\frac{\alpha\left(2-\alpha\right)\left(n-\alpha\right)\left(n-2-\alpha\right)a_n}{b_n + \left(\alpha-1\right)\left(2-\alpha\right)^2 n\left(n-2-\alpha\right)a_n}\right)^{\frac{1}{2-\alpha}}$$

The Paneitz-Branson operator P expresses as

$$P(u) = \Delta_g^2 u - div_g\left(\widetilde{\alpha}\right)du + \widetilde{\beta}u.$$

Given a smooth positive function f on M, the problem is to find a metric $\widetilde{g}$ conformal to the metric g whose Q-curvature is f.

If $\widetilde{g} = u^{\frac{8}{n-4}} g$, u will be the solution of the following equation

$$\Delta_g^2 u - div_g\left(\widetilde{\alpha}\right) du + \widetilde{\beta} u = f u^{N-2} u \tag{6.6}$$

with $N = \frac{2n}{n-4}$.

For any $u \in H_2(M)$, we let

$$J(u) = \int_M \left(\Delta_g u\right)^2 dv_g + \int_M \widetilde{\alpha} \left|\nabla u\right|_g^2 dv_g + \int_M \widetilde{\beta} u^2 dv_g$$

$$B = \left\{ u \in H_2(M) : \int_M f u^N dv_g = 1 \right\}$$

obviously $B \neq \phi$.

Theorem 8. *Let (M^n, h) be a compact flat n-manifold with $n \geqslant 6$ and let $g = Ah$ where $A = e^{-\rho^{2-\rho}}$, $\rho < \inf\{\rho_i : i = 1, 2, 3\}$and $0 < \alpha < 2$. Let f be a C^∞ positive function on M such that*

$$Q(M) < \sup_{x \in M} f(x) K(n, 2).$$

Then there exists a metric $\widetilde{g}$ conformal to g such that f is the Q-curvature of the manifold $(M, \widetilde{g})$.

Proof. Put

$$Q(M) = \inf_{u \in H_2(M) - \{0\}} Q(u) = \inf_{u \in A} J(u).$$

Let u be the solution previously constructed of the equation (6.6), u is at least of class $C^{4,\alpha}$. Let v be a solution of the equation

$$\Delta v + \frac{\widetilde{\alpha}}{2} v = \left| \Delta u + \frac{\widetilde{\alpha}}{2} u \right|$$

where $\widetilde{\alpha}$ is given by (6.1). v exists, since $\widetilde{\alpha} > 0$.

Put

$$w = v \pm u$$

so

$$\Delta w + \frac{\widetilde{\alpha}}{2} w = \left| \Delta u + \frac{\widetilde{\alpha}}{2} u \right| \pm \left(\Delta u + \frac{\widetilde{\alpha}}{2} u \right) \geqslant 0.$$

Hence we get

$$-\Delta(-w) - \frac{\widetilde{\alpha}}{2}(-w) \geq 0 \tag{6.7}$$

By the maximum principle, $-w = u - v$ reaches a maximum $M \geq 0$, which will be constant, but this is excluded since the equation (6.7) implies that $M = 0$. Necessarily, we get $w = v - u > 0$ i.e. $u < v$. In the case $w = u + v$, we obtain $v > -u$. Hence

$$v > |u| \geqslant 0.$$

On the hand, we have

$$\int_M f v^N dv_g \geqslant \int_M f u^N dv_g = 1,$$

we let $0 < k < 1$, such that

$$\int_M f\left(kv\right)^N dv_g = 1$$

and put $\widehat{v}=kv$, then $\widehat{v}>0$ and satisfies $\int_M f\widehat{v}^N dv_g=1$.

Independently, we have

$$\int_M \left((\Delta_g v)^2+\widetilde{\alpha}\,|\nabla v|_g^2+\frac{\widetilde{\alpha}^2}{4}v^2\right)dv_g=\int_M \left((\Delta_g u)^2+\widetilde{\alpha}\,|\nabla u|_g^2+\frac{\widetilde{\alpha}^2}{4}u^2\right)dv_g$$
$$+\frac{1}{2}\int_M \left(u^2-v^2\right)\Delta_g\widetilde{\alpha}dv_g$$

and evaluating

$$S=\int_M \left((\Delta_g \widehat{v})^2+\widetilde{\alpha}\,|\nabla \widehat{v}|_g^2+\widetilde{\beta}\widehat{v}^2\right)dv_g-Q\,(M)$$
$$=k^2\int_M \left((\Delta_g u)^2+\widetilde{\alpha}\,|\nabla u|_g^2+\widetilde{\beta}u^2\right)dv_g+k^2\int_M (\widetilde{\beta}-\frac{\widetilde{\alpha}^2}{4})\left(v^2-u^2\right)dv_g$$
$$+\frac{1}{2}k^2\int_M \left(u^2-v^2\right)\Delta_g\widetilde{\alpha}dv_g-Q(M)$$
$$=\left(k^2-1\right)Q(M)+k^2\int_M (\widetilde{\beta}-\frac{\widetilde{\alpha}^2}{4})\left(v^2-u^2\right)dv_g+\frac{1}{2}k^2\int_M \left(u^2-v^2\right)\Delta\widetilde{\alpha}dv_g.$$

If $1<\alpha<2$ and $\rho<\left(\frac{\alpha(n-\alpha)}{n(2-\alpha)(\alpha-1)}\right)^{\frac{1}{2-\alpha}}$ then by (6.5), we have

$$\Delta\widetilde{\alpha}>0.$$

Since $k^2-1<0$, $Q(M)\geqslant 0$ and $(\overline{\alpha}-\frac{\alpha^2}{4})\left(v^2-u^2\right)\leq 0$, we get

$$S\leq 0.$$

□

7. Proof of theorems 1 and 3

Let $P\in M$ such that $f(P)$ is the maximum of f on M and the metric is of class C^∞ on the ball $B_{2\varepsilon}(P)$ where $0<2\varepsilon<\delta$ and δ is the injectivity radius.

Consider the function

$$\varphi_\epsilon(r)=\frac{\eta(r)}{(r^2+\epsilon^2)^{\frac{n-4}{2}}}$$

where $\eta(\rho)$ is a C^∞ function on M given by

$$\eta(r)=\begin{cases} 1 \text{ on } B_\epsilon\,(P) \\ 0 \quad \text{on } B_{2\epsilon}\,(P)\end{cases}.$$

For $n>6$, by Hölder's inequality, we get

$$B=\int_M a(x)\,|\nabla\varphi_\epsilon(r)|^2\,dv_g\leq\left(\int_M |a(x)|^p\,dv_g\right)^{\frac{1}{p}}\left(\int_M |\nabla\varphi_\epsilon(r)|^{\frac{2p}{p-1}}\,dv_g\right)^{1-\frac{1}{p}}$$

and taking account of

$$|\nabla\varphi_\epsilon(r)|=(n-4)\frac{r}{(\epsilon^2+r^2)^{\frac{n-2}{2}}}$$

and

$$dv_g=(1-\frac{1}{6}R_{ij}x^ix^j)+o(r^2)$$

we get

$$B' = \int_M |\nabla\varphi_\epsilon(r)|^{\frac{2p}{p-1}} dv_g = (n-4)^{\frac{2p}{p-1}} \omega_{n-1} \int_0^\delta (1-\frac{S_g}{6n}r^2)\frac{r^{\frac{2p}{p-1}+n-1}}{(\epsilon^2+r^2)^{(n-2)\frac{p}{p-1}}} dr + o\left(\epsilon^3\right)$$

if we put

$$t = \left(\frac{r}{\epsilon}\right)^2$$

$$B' = \epsilon^{-n+4+\frac{2p}{p-1}} (n-4)^{\frac{2p}{p-1}} \omega_{n-1} \int_0^{\frac{\delta^2}{\epsilon^2}} (1-\frac{S_g}{6n}\epsilon^2 t)\frac{t^{\frac{p}{p-1}+\frac{n-2}{2}}}{(1+t)^{(n-2)\frac{p}{p-1}}} dt +$$

$$= \epsilon^{-n+4+\frac{2p}{p-1}} (n-4)^{\frac{2p}{p-1}} \omega_{n-1} \left(I_{(n-2)\frac{p}{p-1}}^{\frac{p}{p-1}+\frac{n-2}{2}} - \epsilon^2 \frac{S_g}{6n} I_{(n-2)\frac{p}{p-1}}^{\frac{p}{p-1}+\frac{n-2}{2}+1} \right)$$

and taking account of

$$I_p^{q+1} = \frac{q+1}{p-q-2} I_p^q$$

we obtain

$$B'^{1-\frac{1}{p}} = \epsilon^{-(n-4)+2+\frac{n-4}{p}} (n-4)^2 \omega_{n-1}^{1-\frac{1}{p}} \left(I_{(n-2)\frac{p}{p-1}}^{\frac{p}{p-1}+\frac{n}{2}-1}\right)^{1-\frac{1}{p}} \left(1-\epsilon^2\frac{S_g}{6n}\frac{\frac{n}{2}+\frac{p}{p-1}}{(n-3)\frac{p}{p-1}-\frac{n}{2}-1} + o\left(\epsilon^2\right)\right)^{1-\frac{1}{p}}.$$

Hölder's inequality leads to

$$\begin{aligned} C &= \int_M b(x) \left(\varphi_\epsilon(r)\right)^2 dv_g \\ &\leq \left(\int_M |b(x)|^p \, dv_g\right)^{\frac{1}{p}} \left(\int_0^\delta \varphi_\epsilon(r)^{\frac{2p}{p-1}} dv_g\right)^{1-\frac{1}{p}}; \end{aligned}$$

$$\begin{aligned} &\int_0^\delta \varphi_\epsilon(r)^{\frac{2p}{p-1}} dv_g \\ &= \omega_{n-1}\int_0^\delta \frac{r^{n-1}}{(\epsilon^2+r^2)^{(n-4)\frac{p}{p-1}}}\left(1-\frac{S_g}{6n}r^2\right) dr \\ &= \frac{1}{2}\epsilon^{-2(n-4)\frac{p}{p-1}+n}\omega_{n-1} I_{(n-4)\frac{p}{p-1}}^{\frac{n}{2}-1}\left(1-\frac{1}{(n-4)\frac{2p}{p-1}-n}\frac{S_g}{6}\epsilon^2 + o\left(\epsilon^2\right)\right) \end{aligned}$$

Hence

$$C \leq \frac{1}{2}\epsilon^{-2(n-4)+n\frac{p-1}{p}} \omega_{n-1}^{\frac{p-1}{p}} \|b\|_p \left(I_{(n-4)\frac{p}{p-1}}^{\frac{n}{2}-1}\right)^{\frac{p-1}{p}} + o\left(\epsilon\right).$$

The expansion of $\int_{B_\delta(P)} f(x)\varphi_\epsilon^N(r)\, dv_g$ has been computed in [**4**] and is given by

$$\int_{B_\delta(P)} f(x)\varphi_\epsilon^N(r)\, dv_g = \frac{\omega_{n-1}I_n^{\frac{n}{2}-1}}{2}\epsilon^{-n}\left(f(P)-\frac{\epsilon^2}{n-2}\left(\frac{\Delta f}{2}+\frac{f(P)S_g}{6}\right)+o\left(\epsilon^2\right)\right)$$

hence

$$\left(\int_{B_\delta(P)} f(x)\varphi_\epsilon^N(r)\, dv_g\right)^{-\frac{2}{N}} =$$

$$= \left(\frac{\omega_{n-1}I_n^{\frac{n}{2}-1}}{2}\epsilon^{-n}f(P)\right)^{-\frac{n-4}{n}}\left(1+\epsilon^2\frac{n-4}{n(n-2)}\left[\frac{\Delta f}{2f(P)}+\frac{S_g}{6}\right]+o\left(\epsilon^2\right)\right).$$

and

$$\|\varphi_\epsilon\|_N^{-2} = \frac{2^{\frac{n-4}{n}}\epsilon^{n-4}}{\left(I_n^{\frac{n}{2}-1}\omega_{n-1}f(P)\right)^{\frac{n-4}{n}}}\left(1+\epsilon^2\frac{n-4}{n\,(n-2)}\left(\frac{\Delta f}{2f(P)}+\frac{S_g}{6}\right)+o\left(\epsilon^2\right)\right)$$

Also, we have

$$\int_{B_\delta(P)}\left(\Delta\varphi_\epsilon\right)^2 dv_g = \frac{\omega_{n-1}n\,(n-4)\left(n^2-4\right)}{2}\epsilon^{-(n-4)}I_n^{\frac{n}{2}-1}$$
$$\times\left(1-\epsilon^2 S_g\left(\frac{\left(n^2+4\right)(n-4)}{6\,(n-6)\,n(n^2-4)}+\frac{4\,(n-1)}{3n\,(n-6)\,(n+2)}\right)\right)+o\left(\epsilon^2\right)$$

Now by the formula

$$I_p^q = \frac{\Gamma\left(q+1\right)\Gamma\left(p-q-1\right)}{\Gamma\left(p\right)}$$

we get

$$I_n^{\frac{n}{2}-1} = \frac{\Gamma\left(\frac{n}{2}\right)^2}{\Gamma\left(n\right)}$$

and

$$I_{(n-2)\frac{p}{p-1}}^{\frac{n}{2}-1+\frac{p}{p-1}} = \frac{\Gamma\left(\frac{n}{2}+\frac{p}{p-1}\right)\Gamma\left((n-3)\frac{p}{p-1}-\frac{n}{2}\right)}{\Gamma\left((n-2)\frac{p}{p-1}\right)}$$

Resuming, we have

$$J\left(\varphi_\epsilon\right) \le \frac{2^{\frac{n-4}{n}}\epsilon^{n-4}}{\left(I_n^{\frac{n}{2}-1}\omega_{n-1}f(P)\right)^{\frac{n-4}{n}}}\left(1+\epsilon^2\frac{n-4}{n\,(n-2)}\left(\frac{\Delta f\left(P\right)}{2f(P)}+\frac{S_g}{6}\right)+o\left(\epsilon^2\right)\right)\times$$
$$\frac{\omega_{n-1}n\,(n-4)\left(n^2-4\right)}{2}\epsilon^{-(n-4)}I_n^{\frac{n}{2}-1}\left[1-\epsilon^2\frac{n^2+4n-20}{6\,(n-6)\left(n^2-4\right)}S_g+o\left(\epsilon^2\right)\right.$$
$$+\epsilon^{+2+\frac{n-4}{p}}\frac{2n\left(n^2-4\right)}{(n-4)}\|a\|_p\,\omega_{n-1}^{-\frac{1}{p}}\left(I_{(n-2)\frac{p}{p-1}}^{\frac{p}{p-1}+\frac{n}{2}-1}\right)^{1-\frac{1}{p}}\left(I_n^{\frac{n}{2}-1}\right)^{-1}\times$$
$$\left.\left(1-\epsilon^2\frac{S_g}{6n}\frac{\frac{n}{2}+\frac{p}{p-1}}{(n-3)\frac{p}{p-1}-\frac{n}{2}-1}+o\left(\epsilon^2\right)\right)^{1-\frac{1}{p}}\right]$$
$$=\omega_{n-1}n\,(n-4)\left(n^2-4\right)\left(\frac{I_n^{\frac{n}{2}-1}\omega_{n-1}}{2}\right)^{\frac{4}{n}}f(P)^{-\frac{2}{N}}$$
$$\times\left[1-\epsilon^2\left\{\frac{n^2+4n-20}{6\,(n-6)\left(n^2-4\right)}S_g-\frac{n-4}{2n\,(n-2)}\frac{\Delta f(P)}{f(P)}\right\}\right]+o\left(\epsilon^2\right).$$

Recall the value of the best Sobolev constant,

$$K_2^{-2} = n\,(n+2)\,(n-2)\,(n-4)\left(\frac{I_n^{\frac{n}{2}-1}\omega_{n-1}}{2}\right)^{\frac{4}{n}}$$

so if

$$\frac{4\left(n^2-2n-4\right)}{(n-6)\,(n-2)\,(n+2)}S_g-\frac{\Delta f(P)}{f(P)}>0$$

we get

$$Q(M) < K_2^{-2} f(P)^{-\frac{2}{N}} \qquad \text{as} \qquad \epsilon \to 0^+.$$

For $n = 6$, direct computations give

$$\int_M (\Delta\varphi_\epsilon)^2 \, dv_g = K_2^{-2} f(P)^{-\frac{1}{3}} - 4\epsilon^2 \log\left(\frac{1}{\epsilon^2}\right) \left(\frac{\omega_{n-1}}{2}\right)^{\frac{3}{4}} \left(f(P) I_n^{\frac{n}{2}-1}\right)^{-\frac{1}{3}} \frac{S_g(P)}{3} + o\left(\epsilon^2\right)$$

so

$$Q(M) < K_2^{-2} f(P)^{-\frac{1}{3}}$$

if $S_g(P) > 0$ as $\epsilon \to 0^+$.

References

[1] Aubin, T., Some nonlinear problems in Riemannian geometry, Springer (1998).

[2] Benalili, M., Boughazi, H., On the second Paneitz- Branson invariant. Houston Journal of Mathematics, vol., 36, 2, 2010, 393-420.

[3] Benalili, M., Existence and multiplicity of solutions to elliptic equations of fourth order on compact manifolds. Dynamics of PDE, vol.6, 3 (2009), 203-225.

[4] Benalili, M., Existence and multiplicity of solutions to fourth order elliptic equations with critical exponent on compact manifolds, Bull. Belg. Math. Soc. Simon Stevin 17 (2010).

[5] Benalili, M., Kamel, T., Nonlinear elliptic fourth order equations existence and multiplicity results (preprint, Tlemcen 2010)

[6] Branson, T.P., Group representation arising from Lorentz conformal geometry. J. Funct. Anal. 74, 1987, 199-291.

[7] Caraffa,D. Equations elliptiques du quatrième ordre avec exposants critiques sur les variétés riemanniennes compactes. J. Math. Pures Appl., 80, 9 (2001), 941-960.

[8] Chang S.Y.A., On a Paneitz operator, a fourth order differential operator in conformal geometry, Harmonic analysis and partial differential equations, Essays in Honor of Alberto P. Calderon, Eds. M. Christ, C. Kening and C. Sadorsky, Chicago Lectures in Mathematics, 1999, 127-150.

[9] Chang S.Y.A., Yang P.C., On a fourth order curvature invariant, Comp. Math. 237, Spectral Problems in Geometry and Arithmetics, Edition T. Branson, AMS, 1999, 9-28.

[10] Djadli, Z. Hebey, E. Ledoux, M., Paneitz type operators and applications. Duke Math. J. 104, 2000, 129-169.

[11] Djadli, Z., Malchiodi, A., Ould Ahmedou, M., Prescribing a fourth order conformal invariant on the standard sphere. I. A perturbation result, Commun. contemp. Math. , 4., 2002, 375-408.

[12] Djadli, Z., Malchiodi, A., Ould Ahmedou, M., Prescribing a fourth order conformal invariant on the standard sphere. II. Bow up analysis and applications, Ann. Sc. Norm. Super. Pisa Cl. Sci., 1, 387-434, 2002.

[13] Edmunds, D.E., Furtunato, F., Janelli, E., Critical exponents, critical dimensions and biharmonic operators, Arch. Rational Mech.Anal.1990, 269-289.

[14] Giraud, G., Sur le problème de Dirichlet généralisé, Ann. Sc. Ecole Norm. Sup., 46, 1929, 131-235.

[15] Hebey, E., Sharp Sobolev inequalities of second order, J. Geom. Anal. 13, 1, (2003), 145-162.

[16] Hebey, E., Robert, F., Coercivity and Struwe's compactness for Paneitz type operators with constant coefficients, Cal. Var. Partial Differential Equations, 13, 2001, 491-517.

[17] Madani, F., Le problème de Yamabé avec singularités, ArXix: 1717v1 [mathAP] 10 Apr 2008.

[18] S. Paneitz, A quatric conformally covariant differential operator for arbitrary peudo-Riemannian manifolds, SIGMA, 4, (2008).

[Faculty of Sciences, Dept. of Mathematics. BP 110, University Abou Bekr Belkaïd Tlemcen. Algeria.

E-mail address: m_benalili@mail.univ-tlemcen.dz